

# Contributions to Intertemporal Models in Financial Economics

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## Abstract:

Investors in financial markets face several restrictions apart from wealth constraints. The first attempt to understand these restrictions in a general competitive equilibrium framework can be traced back to Radner (1972). Here these restrictions are assumed to be given exogenously, as first modeled by Siconolfi (1986). Portfolio sets given by closed and convex sets containing zero are the most general form of portfolio restrictions. Such sets can accommodate most restrictions that investors actually face in financial markets, see Elsinger and Summer (2000). The traditional Arrow-Debreu general equilibrium models can then be extended to a more realistic setting. Following Angeloni and Cornet (2006), this extension of the Arrow-Debreu model in the multi-period setting with restricted participation is established.

Once investors are assumed to face such portfolio restrictions, there is a need to differentiate individual arbitrage opportunities from those at the aggregate level. In the special case where the portfolio sets linear subspace (for example investors invest only through some mutual funds) this difference in the notion of arbitrage at the individual level and the aggregate level is characterized. Extending the 2-date result of Hens *et al.*, we show that *generically* there will be some arbitrage opportunities that remain unexploited at the aggregate level.

The existence of a *general financial equilibrium*, an equilibrium in all markets (commodities and financial assets) has been extensively studied in 2-date, multi-period and infinite horizon models. With such general portfolio con-

straints, using an approach that dates back to Cass 1984, we look for *arbitrage-free* asset prices at the aggregate level that are also equilibrium asset prices. For this approach, we present a condition on the space of income transfers which makes the existence result in this work very general and the previous results in this area turn out to be special cases.

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# Chapter 1

## Introduction

The pioneering work of Arrow [4] and Debreu [19], [20] in the 1950s, has had an enduring effect on the study of financial aspects of the economy in a general equilibrium framework. One of their key contributions was to introduce time and uncertainty into general equilibrium models with the use of a date-event tree. In this set up commodities are available in each date-event (spot) and can be traded at the initial date. A classic result of Arrow in this work was to show that when sufficiently many *Arrow securities* (a contract that pays one unit of account if a particular date-event occurs and nothing otherwise), can be traded at the initial date-event, to transfer income across all spots, and commodities are traded at each spot, then an equilibrium in such a financial economy is equivalent to the Arrow-Debreu equilibrium in contingent commodities. The inclusion of the financial aspects of the economy in this manner enables a more realistic model where trade happens sequentially.

When sufficiently many financial assets (like Arrow securities) are available so that all the agents are able to transfer income across all spots, then markets are said to be *complete*. *Incomplete* markets is then the phenomenon where the financial assets available for trade are not sufficient to transfer income across



all spots and hence agents are limited in their risk sharing possibilities.

This observation that markets may be incomplete led to the development of the growth in the literature on general equilibrium with incomplete markets (GEI). One of the early contributions in this respect is that of Radner [58] in 1972. In Radner's model, assets are contracts that promise to deliver one unit of a contingent commodity upon the realization of a date event. On the one hand, Radner shows that unlike the Arrow-Debreu models, the possibility of trading commodity futures for every contingency is sufficient to enable income transfers across all spots.

On the other hand, Radner also assumes that short-sales of these contracts are limited for every agent. This assumption is a driving force in his proof of the existence of a general financial equilibrium. This can be seen as the first attempt to incorporate the idea that agents may be restricted in their asset market participation.

However, Hart [44], used some disturbing yet insightful counter-examples to demonstrate some of the weaknesses of Radner's concept of equilibrium. Firstly, he showed that existence of such an equilibrium cannot be proved under the standard Arrow-Debreu assumptions. The reason for this non-existence is, when the asset returns are price dependent, the market sub-space may not be continuous in the spot prices which may lead to discontinuous demand functions. This causes a failure of the existence of a Radner equilibrium. In the case of incomplete markets an equilibrium may not be pareto optimal. Hence in these models a notion of *Constrained Pareto Optimality* is defined. In addition, Hart in his 1975 paper, using a three date model, also showed that, the intuition that, introducing more assets may lead to equilibrium that is

pareto superior, is false. Similar results were also obtained by Bhattacharya [8] in a two date model.

Two possible ways emerged, to avoid such discontinuities in the demand functions. The first was to restrict attention to assets for which the return matrix cannot change rank. As for instance restricting attention to assets that payoff in units of account (Cass [13], Werner [65]) or in a numeraire asset (Geanakoplos-Polemarchakis [34] and Magill and Shafer [51]). The second approach involves, the techniques using differential topology, introduced by Debreu [21] into economics. This has become the standard methodology to demonstrate existence, constrained efficiency and indeterminacy of equilibria in incomplete markets. In this approach, economies with general return structures are considered and the procedure involved is to show that the sets on which the required properties do or do not hold, are negligible.

Cass [13] in his famous 1984 working paper published much later in 2006 [15], makes several important observations, two of which are directly related to the work in this thesis. Firstly, Cass makes the following observation:

*“... one of the most fascinating outcomes of the Arrow-Debreu-McKenzie development of the solution to the equilibrium existence problem was the explicit recognition of the singular importance of some minimal commonality between households. This specific aspect of my model deserves further serious study (requiring subtle analysis of the interplay between endowments, preferences and financial opportunities.”*

The second observation, to which this thesis attempts an answer, is related to the case of restricted participation. Restricted Participation models are those where, in addition to the usual budget constraints, each agent faces some

financial constraints in term of the portfolio restrictions. Here Cass conjectures that given that each agent is restricted to some subset of portfolios (closed, convex and containing zero), if some agent is able to access all the portfolios available to every other agent then existence is ensured.

Several contributions have been made in the literature to prove existence of a general financial equilibrium in the case of incomplete markets, using the two approaches mentioned above. In the two-date (one period) case the major contributions include Cass ([13], [15]), Werner ([65], [66]), Magill and Shafer ([51] [52]) and Geanakoplos and Polemarchakis [34], among others. Bich and Cornet [9] have proved the existence of equilibrium in a two-date model when agents may have non-transitive preferences.

## 1.1 Extensions to the Arrow-Debreu model

The multi-period extension of the GEI model requires a systematic treatment of the evolution of time and uncertainty. Following Debreu's [20] classic work, *Theory of Value*, multi-period models have been posed in the event-tree framework alluded to earlier. In this framework information unfolds over time. Radner [58] formalized this model for a finite period setting. The book by Magill and Quinzii [53] and the paper by Angeloni and Cornet [1] give a good introduction to models in this framework. Existence of equilibrium in the multi-period case was first proved by Duffie and Schafer [26]. Other existence results have followed since then, Florenzano and Gourdel [29], Levine and Zame [49] and Monteiro and Pascoa [55], among others.

There have been several other extensions, departing from the implicit and explicit assumptions of the Arrow-Debreu framework and the GEI models.

The case of Asymmetric information in GEI models has been studied by Boisset and Cornet [10]. The possibility of default was introduced into these class of models by Dubey, Geanakoplos and Zame [23], Dubey, Geanakoplos and Shubik [22] and Zame [68]. Araujo, Pascoa and Orillo [2] and Araujo, Pascoa and Martinez [3], have introduced collateral as an institution to prevent *Ponzi Games* and preclude default.

## 1.2 Restricted participation

The extension that this thesis deals with, is that stemming from the comments and observations in Cass' 1984 working paper, the case of restricted participation.

There have been two distinct approaches to incorporate restricted participation into these models. The first approach assumes that these restrictions arise due to institutional reasons and hence are part of the primitives of the model. This would help to cover institutional details like, transactions costs, short-sales constraints, margin requirements, collateral requirements, market frictions due to bid-ask spreads and proportional taxes, capital adequacy requirements and target ratios. Luttmer [50] and Elsinger and Summer [27] describe how such institutional factors can be accommodated in the model by assuming that agents have closed, convex portfolio sets containing zero.

Siconolfi [61]; Balasko, Cass and Siconolfi [6]; Benveniste and Ketterer [7] and Polemarchakis and Siconolfi [57] consider linear restrictions on portfolio sets. Cass, Siconolfi and Villanacci [16] and Villanacci et al. [63] work with differentiable economies to study the nature of equilibrium. They consider portfolio sets described by differentiable functions. Considering closed,

convex portfolio sets containing zero would be the most general types of primitive restrictions that can be considered. As mentioned earlier these types of portfolio sets would help to cover a wide variety of institutional restrictions mentioned earlier. Siconolfi [61] and more recently, Angeloni and Cornet [1]; DaRocha and Triki [54] and Hahn and Won [41] study the issue of existence of equilibrium under such general portfolio restrictions.

The other, relatively less explored, approach is one where these portfolio constraints are determined endogenously. Here each agents portfolio constraint may depend on the equilibrium commodity prices and equilibrium asset prices and other variables that will be determined at equilibrium. Carosi [11] and Carosi and Villanacci [12] study these types of endogenous restrictions in a differentiable economy.

### 1.3 The absence of arbitrage

The absence of arbitrage in financial markets is in essence the *no free lunch* postulate. It is not possible to obtain something without giving up something else. More formally, given asset prices and a set of possible portfolios does not offer arbitrage opportunities if it does not contain any portfolio that yields non-negative net returns in any date-event and strictly positive net returns in at least one date-event.

Models in which, there is no restricted participation, every agent's portfolio set is unconstrained and thus an asset prices that does not offer arbitrage opportunities to the market as a whole will also deny arbitrage opportunities to each individual agent and vice versa. However once, we impose the realistic idea of constrained portfolio sets, this notion of absence of arbitrage at the

aggregate level differ considerably from that at the individual level. Angeloni and Cornet [1] give a definition of no-arbitrage at the aggregate level and at the individual level. The notion of arbitrage and its absence in this thesis will follow this definition.

Even without trading frictions like, short-sales constraints or transactions costs, agents may restrict their attention to only a subset of assets. This can be seen as arising due the fact the agent are limited in their attention span and information processing capabilities. With these assumptions, in a two-date model, Hens et al. [47] show that there is a considerable difference in the arbitrage possibilities at the aggregate level as compared to those at the individual level. In fact they show that at some asset prices, even if none of the agents can find an arbitrage opportunities within their portfolio sets, there may be some opportunities left unexploited at the aggregate (market) level. The first contribution of this thesis, is to extend this result to the multi-period (finite) case. This extension would require us to address two issues in particular. Firstly, the presence of long-lived in asset in the multi-period setting, gives rise to the possibility of the deepening of mispricing which may make arbitrage more expensive. This may naturally leave arbitrage opportunities unexploited at the aggregate level. However in the rational expectations framework assumed here, with no other market frictions, this possibility is supposed away. The second issue, is to suitably modify the total returns matrix to allow for frequent retrading of long-lived assets. This modification is done in a similar spirit to Angeloni and Cornet [1].

## 1.4 General financial equilibrium

Taking the constraints on individual portfolio sets as a primitive in the description of a consumer along with her initial endowments and preference relations, we have a model of restricted participation with exogenous restricted participation. A general financial equilibrium in such a model is a list of consumption allocations, portfolio allocations and commodity and asset prices at which (i) each consumer chooses a consumption bundle and a portfolio to maximize her preferences under her budget restrictions, (ii) commodity markets clear and (iii) asset markets clear.

An interesting result arising from the paper by Cass [13] in 1984 which he also recalls in the paper by Balasko, Cass and Siconolfi [6], is that, with no restricted participation, asset prices that do not offer arbitrage opportunities at the aggregate level will also be equilibrium asset prices. This result has also been explored by Duffie [24] and Florenzano and Gourdel [29]. More recently in a working paper, Da-Rocha and Triki [54] have explored this characterization with very general portfolio sets.

The existence proof of Cass [13] and Magill and Shafer [51] relies heavily on the approach that has now come to be known as the ‘Cass Trick’ [15]. In this approach one agent is assumed to be able to accommodate all the excess demand in the asset market. In other words, this agent is assumed to be unconstrained and hence behaves as in an Arrow-Debreu economy. Another weaker assumption is that of Angeloni and Cornet [1], where one agent is assumed to have zero in the interior of her portfolio set. This assumption is again needed for the use of the Cass Trick. More recently, Da-Rocha and Triki [54] have outlined a procedure to avoid, this ad hoc assumption causing the

treatment of agent's constraints in an asymmetry manner. Hahn and Won [41] also in a recent working paper prove existence without the use of the Cass Trick, albeit with monotone preferences and a more involved notion of absence of arbitrage.

The second major contribution of this thesis is to utilize the approach of Da-Rocha and Triki [54], alluded to earlier, to prove the existence of a general financial equilibrium. In this existence result consumers have very general preference relations. In showing this characterization result of no-arbitrage prices with equilibrium prices, a crucial assumption on the possibility of income transfers in the aggregate level is needed. Here this condition is called the *transfer space condition*. The result here differs from that of Da-Rocha and Triki, as we suppose a more general notion of absence of arbitrage and more general transfer space condition.

Chapter 2, outlines the general model of restricted asset market participation where the restrictions are given exogenously. Chapter 3 presents and discusses the first main result of this thesis alluded to earlier in Section 1.3. Chapter 4 presents and discusses the main existence result mentioned in Section 1.4. This chapter also elucidates how many of the previous results in this area can be obtained as corollaries of this central result. Chapter 5 concludes this thesis and discusses some future directions for research.



## Chapter 2

### The Model

#### 2.1 Time and uncertainty

In order to describe a model of the multi-period financial exchange economy, we need to begin with the evolution of time and uncertainty. Motivated by the ideas in Chapter 7 of Debreu [20], the following model of time and uncertainty is well known in the literature. For a detailed description see the two equivalent formulations by Magill and Quinzii [53] and Angeloni and Cornet [1]. The exposition here will follow the latter, which explicitly models the type of restricted participation that is of interest in this thesis.

We<sup>1</sup> consider a multiperiod exchange economy with  $(\mathbb{T} + 1)$  dates,  $t \in \mathcal{T} :=$

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<sup>1</sup>In this paper, we shall use the following notations. A  $(\mathbb{D} \times \mathbb{J})$ -matrix  $A$  is an element of  $\mathbb{R}^{\mathbb{D} \times \mathbb{J}}$ , with entries  $(a(\xi, j))_{\xi \in \mathbb{D}, j \in \mathbb{J}}$ ; we denote by  $A(\xi) \in \mathbb{R}^{\mathbb{J}}$  the  $\xi$ -th row of  $A$  and by  $A(j) \in \mathbb{R}^{\mathbb{D}}$  the  $j$ -th column of  $A$ . We recall that the transpose of  $A$  is the unique  $(\mathbb{J} \times \mathbb{D})$ -matrix  ${}^tA$  satisfying  $(Ax) \bullet_{\mathbb{D}} y = x \bullet_{\mathbb{J}} ({}^tAy)$ , for every  $x \in \mathbb{R}^{\mathbb{J}}$ ,  $y \in \mathbb{R}^{\mathbb{D}}$ , where  $\bullet_{\mathbb{D}}$  [resp.  $\bullet_{\mathbb{J}}$ ] denotes the usual scalar product in  $\mathbb{R}^{\mathbb{D}}$  [resp.  $\mathbb{R}^{\mathbb{J}}$ ]. We shall denote by  $\text{rank} A$  the rank of the matrix  $A$ . For every subsets  $\tilde{\mathbb{D}} \subset \mathbb{D}$  and  $\tilde{\mathbb{J}} \subset \mathbb{J}$ , the  $(\tilde{\mathbb{D}} \times \tilde{\mathbb{J}})$ -sub-matrix of  $A$  is the  $(\tilde{\mathbb{D}} \times \tilde{\mathbb{J}})$ -matrix  $\tilde{A}$  with entries  $\tilde{a}(\xi, j) = a(\xi, j)$  for every  $(\xi, j) \in \tilde{\mathbb{D}} \times \tilde{\mathbb{J}}$ . Let  $x, y$  be in  $\mathbb{R}^n$ ; we shall use the notation  $x \geq y$  (resp.  $x \gg y$ ) if  $x_h \geq y_h$  (resp.  $x_h \gg y_h$ ) for every  $h = 1, \dots, n$  and we let  $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x \geq 0\}$ ,  $\mathbb{R}_{++}^n = \{x \in \mathbb{R}^n : x \gg 0\}$ . We shall also

$\{0, \dots, \mathbb{T}\}$ , and a finite set of agents  $\mathcal{I} = \{1, \dots, \mathbb{I}\}$ . The stochastic structure of the model is described by a finite event-tree  $\mathcal{D}$  of length  $\mathbb{T}$  and we shall essentially use the same model as Angeloni and Cornet [1], (we refer to [53] for an equivalent presentation with information partitions). The set  $\mathcal{D}_t$  denotes the nodes (also called date-events) that could occur at date  $t$  and the family  $(\mathcal{D}_t)_{t \in \mathcal{T}}$  defines a partition of the set  $\mathcal{D}$ ; for each  $\xi \in \mathcal{D}$  we denote by  $t(\xi)$  the unique  $t \in \mathcal{T}$  such that  $\xi \in \mathcal{D}_t$ . Also we denote the cardinality of the set  $\mathcal{D}$  by  $\mathbb{D}$ .

At each date  $t \neq \mathbb{T}$ , there is an a priori uncertainty about which node will prevail in the next date. There is a unique non-stochastic event occurring at date  $t = 0$ , which is denoted  $\xi_0$ , (or simply 0) so  $\mathcal{D}_0 = \{\xi_0\}$ . Finally, every  $\xi \neq \xi_0$  in the event-tree  $\mathcal{D}$  has a unique predecessor denoted  $pr(\xi)$  in  $\mathcal{D}$ . The predecessor mapping  $pr : \mathcal{D} \setminus \{\xi_0\} \longrightarrow \mathcal{D}$  satisfies  $pr(\mathcal{D}_t) = \mathcal{D}_{t-1}$ , for every  $t \neq 0$ . The element  $pr(\xi)$  is called the immediate predecessor of  $\xi$  and is also denoted  $\xi^-$ . For each  $\xi \in \mathcal{D}$ , we let  $\xi^+ = \{\bar{\xi} \in \mathcal{D} : \xi = \bar{\xi}^-\}$  be the set of immediate successors of  $\xi$ ; we notice that the set  $\xi^+$  is nonempty if and only if  $\xi \in \mathcal{D} \setminus \mathcal{D}_T$ .

Moreover, for  $\tau \in \mathcal{T} \setminus \{0\}$  and  $\xi \in \mathcal{D} \setminus \bigcup_{t=0}^{\tau-1} \mathcal{D}_t$  we define, by induction,  $pr^\tau(\xi) = pr(pr^{\tau-1}(\xi))$  and we let the set of (not necessarily immediate) successors and the set of predecessors of  $\xi$  be respectively defined by

$$\mathcal{D}^+(\xi) = \{\xi' \in \mathcal{D} : \exists \tau \in \mathcal{T} \setminus \{0\} \mid \xi = pr^\tau(\xi')\},$$

$$\mathcal{D}^-(\xi) = \{\xi' \in \mathcal{D} : \exists \tau \in \mathcal{T} \setminus \{0\} \mid \xi' = pr^\tau(\xi)\}.$$

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use the notation  $x > y$  if  $x \geq y$  and  $x \neq y$ . We shall denote by  $\|\cdot\|$  the Euclidean norm in the different Euclidean spaces used in this paper and the closed ball centered at  $x \in \mathbb{R}^{\mathbb{L}}$  of radius  $r > 0$  is denoted  $B_L(x, r) := \{y \in \mathbb{R}^{\mathbb{L}} : \|y - x\| \leq r\}$ .

If  $\xi' \in \mathcal{D}^+(\xi)$  [resp.  $\xi' \in \mathcal{D}^+(\xi) \cup \{\xi\}$ ], we shall also use the notation  $\xi' > \xi$  [resp.  $\xi' \geq \xi$ ]. We notice that  $\mathcal{D}^+(\xi)$  is nonempty if and only if  $\xi \notin \mathcal{D}_T$  and  $\mathcal{D}^-(\xi)$  is nonempty if and only if  $\xi \neq \xi_0$ . Moreover, one has  $\xi' \in \mathcal{D}^+(\xi)$  if and only if  $\xi \in \mathcal{D}^-(\xi')$  and similarly  $\xi' \in \xi^+$  if and only if  $\xi = (\xi')^-$ .

### 2.1.1 The stochastic exchange economy

At each node  $\xi \in \mathcal{D}$ , there is a spot market where a finite set  $\mathcal{H} = \{1, \dots, \mathbb{H}\}$  of divisible physical goods is available. We assume that each good does not last for more than one period. In this model, a commodity is a couple  $(h, \xi)$  of a physical good  $h \in \mathcal{H}$  and a node  $\xi \in \mathcal{D}$  at which it will be available, so the commodity space is  $\mathbb{R}^{\mathbb{L}}$ , where  $\mathbb{L} = \mathbb{H} \times \mathbb{D}$ . An element  $x$  in  $\mathbb{R}^{\mathbb{L}}$  is called a *consumption*, that is  $x = (x(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^{\mathbb{L}}$ , where  $x(\xi) = (x(h, \xi))_{h \in \mathcal{H}} \in \mathbb{R}^{\mathbb{H}}$ , for every  $\xi \in \mathcal{D}$ .

We denote by  $p = (p(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^{\mathbb{L}}$  the vector of spot prices and  $p(\xi) = (p(h, \xi))_{h \in \mathcal{H}} \in \mathbb{R}^{\mathbb{H}}$  is called the spot price at node  $\xi$ . The spot price  $p(h, \xi)$  is the price paid, at date  $t(\xi)$ , for the delivery of one unit of the physical good  $h$  at node  $\xi$ . Thus the value of the consumption  $x(\xi)$  at node  $\xi \in \mathcal{D}$  (evaluated in unit of account of node  $\xi$ ) is

$$p(\xi) \bullet_{\mathbb{H}} x(\xi) = \sum_{h \in \mathcal{H}} p(h, \xi) x(h, \xi).$$

Each agent  $i \in \mathcal{I}$  is endowed with a *consumption set*  $X^i \subset \mathbb{R}^{\mathbb{L}}$  which is the set of her possible consumptions. An *allocation* is an element  $x \in \prod_{i \in \mathcal{I}} X^i$ , and we denote by  $x^i$  the consumption of agent  $i$ , that is the projection of  $x$  onto  $X^i$ .

The tastes of each consumer  $i \in \mathcal{I}$  are represented by a *strict preference*

correspondence  $P^i : \prod_{j \in \mathcal{I}} X^j \longrightarrow X^i$ , where  $P^i(x)$  defines the set of consumptions that are strictly preferred by  $i$  to  $x^i$ , that is, given the consumptions  $x^j$  for the other consumers  $j \neq i$ . Thus  $P^i$  represents the tastes of consumer  $i$  but also her behavior under time and uncertainty, in particular her impatience and her attitude towards risk. If consumers' preferences are represented by utility functions  $u^i : X^i \longrightarrow \mathbb{R}$ , for every  $i \in \mathcal{I}$ , the strict preference correspondence is defined by  $P^i(x) = \{\bar{x}^i \in X^i \mid u^i(\bar{x}^i) > u^i(x^i)\}$ .

Finally, at each node  $\xi \in \mathcal{D}$ , every consumer  $i \in \mathcal{I}$  has a *node-endowment*  $e^i(\xi) \in \mathbb{R}^{\mathbb{H}}$  (contingent to the fact that  $\xi$  prevails) and we denote by  $e^i = (e^i(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^{\mathbb{L}}$  her *endowment vector* across the different nodes. The exchange economy  $\mathcal{E}$  can thus be summarized by

$$\mathcal{E} = [\mathcal{D}; \mathcal{H}; \mathcal{I}; (X^i, P^i, e^i)_{i \in \mathcal{I}}].$$

### 2.1.2 The financial structure

We consider finitely many financial assets and we denote by  $\mathcal{J} = \{1, \dots, \mathbb{J}\}$  the set of assets. An asset  $j \in \mathcal{J}$  is a contract, which is issued at a unique node in  $\mathcal{D}$ , denoted by  $\xi(j)$  and called the *emission node* of  $j$ . Each asset  $j$  is bought (or sold) at its emission node  $\xi(j)$  and only yields payoffs at the successor nodes  $\xi'$  of  $\xi(j)$ , that is, for  $\xi' > \xi(j)$ . We denote by  $v(\xi, j)$  the payoff of asset  $j$  at node  $\xi$ . Since we consider only nominal assets this payoff does not depend on the spot prices. For the sake of convenient notations, we shall in fact consider the payoff of asset  $j$  at every node  $\xi \in \mathcal{D}$  and assume that it is zero if  $\xi$  is not a successor of the emission node  $\xi(j)$ . Formally, we assume that  $v(\xi, j) = 0$  if  $\xi \notin \mathcal{D}^+(\xi(j))$ . With the above convention, we notice that every asset has a zero payoff at the initial node, that is  $v(\xi_0, j) = 0$  for

every  $j \in \mathcal{J}$ . Furthermore, every asset  $j$  which is emitted at the terminal date  $\mathbb{T}$  has a zero payoff, that is, if  $\xi(j) \in \mathcal{D}_T$ ,  $v(\xi, j) = 0$  for every  $\xi \in \mathcal{D}$ .

For every consumer  $i \in \mathcal{I}$ , if  $z_j^i > 0$  [resp.  $z_j^i < 0$ ], then  $|z_j^i|$  will denote the quantity of asset  $j \in \mathcal{J}$  bought [resp. sold] by agent  $i$  at the emission node  $\xi(j)$ . The vector  $z^i = (z_j^i)_{j \in \mathcal{J}} \in \mathbb{R}^{\mathbb{J}}$  is called the *portfolio* of agent  $i$ .

We assume that each consumer  $i \in \mathcal{I}$  is endowed with a *portfolio set*  $Z^i \subset \mathbb{R}^{\mathbb{J}}$ , from which agent  $i$  is restricted to choose her portfolios.

The price of asset  $j$  is denoted by  $q_j$  and we recall that it is paid at its emission node  $\xi(j)$ . We let  $q = (q_j)_{j \in \mathcal{J}} \in \mathbb{R}^{\mathbb{J}}$  be the asset price (vector).

**Definition 2.1.1** *A financial asset structure*

$$\mathcal{F} = (\mathcal{J}, (\xi(j), V^j)_{j \in \mathcal{J}}, (Z^i)_{i \in \mathcal{I}})$$

*consists of*

- a set of assets  $\mathcal{J}$ ,
- each asset  $j \in \mathcal{J}$  is defined by a node of issue  $\xi(j) \in \mathcal{D}$  and the vector of payoffs across all nodes  $V^j \in \mathbb{R}^{\mathbb{D}}$

- a collection of portfolio sets  $Z^i \subset \mathbb{R}^{\mathbb{J}}$  for every agent  $i \in \mathcal{I}$ ,

The payoff matrix is given by the  $\mathbb{D} \times \mathbb{J}$  matrix  $V = (v(\xi, j))_{\xi \in \mathcal{D}, j \in \mathcal{J}}$ , and satisfies the condition  $v(\xi, j) = 0$  if  $\xi \notin \mathcal{D}^+(\xi(j))$ .

The full matrix of payoffs  $W_{\mathcal{F}}(q)$  is the  $(\mathbb{D} \times \mathbb{J})$ -matrix with entries

$$w_{\mathcal{F}}(q)(\xi, j) := v(\xi, j) - \delta_{\xi, \xi(j)} q_j,$$

where  $\delta_{\xi, \xi'} = 1$  if  $\xi = \xi'$  and  $\delta_{\xi, \xi'} = 0$  otherwise.

So, for a given portfolio  $z \in \mathbb{R}^{\mathbb{J}}$  (and asset price  $q$ ) the full flow of payoffs

is  $W_{\mathcal{F}}(q)z$  and the (full) financial payoff at node  $\xi$  is

$$\begin{aligned} [W_{\mathcal{F}}(q)z](\xi) &:= W_{\mathcal{F}}(q, \xi) \bullet_{\mathbb{J}} z = \sum_{j \in \mathcal{J}} v(\xi, j) z_j - \sum_{j \in \mathcal{J}} \delta_{\xi, \xi(j)} q_j z_j \\ &= \sum_{\{j \in \mathcal{J} \mid \xi(j) < \xi\}} v(\xi, j) z_j - \sum_{\{j \in \mathcal{J} \mid \xi(j) = \xi\}} q_j z_j, \end{aligned}$$

and we shall extensively use the fact that, for  $\lambda \in \mathbb{R}^{\mathbb{D}}$ , and  $j \in \mathcal{J}$ , one has:

$$\begin{aligned} [{}^t W_{\mathcal{F}}(q)\lambda](j) &= \sum_{\xi \in \mathcal{D}} \lambda(\xi) v(\xi, j) - \sum_{\xi \in \mathcal{D}} \lambda(\xi) \delta_{\xi, \xi(j)} \\ &= \sum_{\xi > \xi(j)} \lambda(\xi) v(\xi, j) - \lambda(\xi(j)) q_j. \end{aligned} \tag{2.1}$$

In the following, when the financial structure  $\mathcal{F}$  remains fixed, while only prices vary, we shall simply denote by  $W(q)$  the full matrix of payoffs. In the case of unconstrained portfolios, namely  $Z^i = \mathbb{R}^{\mathbb{J}}$ , for every  $i \in \mathcal{I}$ , the financial asset structure will be simply denoted by  $\mathcal{F} = (J, (\xi(j), V^j)_{j \in \mathcal{J}})$ .

The model described above will be the basic framework in which all the results and analysis of this thesis will be carried out. In Chapter 3 we consider a special case of this model. The spot consumption choices are one dimensional,  $\mathbb{H} = 1$ , i.e. there is only one good (income) available for consumption at each node. The spot price of this good is normalized to equal 1. Preferences are given by a utility function that is strictly increasing and concave. On the financial side, each agent is restricted to a subset of assets available in the economy. This results in each agent's portfolio set  $Z^i$  being a strict subspace of  $\mathbb{R}^{\mathbb{J}}$ . Chapter 4 on the other hand will consider the general version of this model described above.

## 2.2 Financial equilibrium

We now consider a financial exchange economy, which is defined as the couple of an exchange economy  $\mathcal{E}$  and a financial structure  $\mathcal{F}$ . It can thus be summarized by

$$(\mathcal{E}, \mathcal{F}) := [\mathcal{D}, \mathcal{H}, \mathcal{I}, (X^i, P^i, e^i)_{i \in \mathcal{I}}; \mathcal{J}, (\xi(j), V^j)_{j \in \mathcal{J}}, (Z^i)_{i \in \mathcal{I}}].$$

Given the price  $(p, q) \in \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathbb{J}}$ , the *budget set* of consumer  $i \in \mathcal{I}$ , is the set <sup>2</sup>

$$\begin{aligned} B_{\mathcal{F}}^i(p, q) &= \{(x^i, z^i) \in X^i \times Z^i : \forall \xi \in \mathcal{D}, p(\xi) \bullet_{\mathbb{H}} [x^i(\xi) - e^i(\xi)] \leq [W_{\mathcal{F}}(q)z^i](\xi)\} \\ &= \{(x^i, z^i) \in X^i \times Z^i : p \square (x^i - e^i) \leq W_{\mathcal{F}}(q)z^i\} \end{aligned}$$

When  $\mathcal{F}$  is fixed we can drop the subscript  $\mathcal{F}$  from the the payoff matrix  $W$  and the budget set. We now introduce the equilibrium notion.

**Definition 2.2.1** *An equilibrium of the financial exchange economy  $(\mathcal{E}, \mathcal{F})$  is a list of strategies and prices  $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in (\mathbb{R}^{\mathbb{L}})^{\mathbb{I}} \times (\mathbb{R}^{\mathbb{J}})^{\mathbb{I}} \times \mathbb{R}^{\mathbb{L}} \setminus \{0\} \times \mathbb{R}^{\mathbb{J}}$  such that*

(a) *for every  $i \in \mathcal{I}$ ,  $(\bar{x}^i, \bar{z}^i)$  maximizes the preferences  $P^i$  in the budget set  $B_{\mathcal{F}}^i(\bar{p}, \bar{q})$ , in the sense that*

$$(\bar{x}^i, \bar{z}^i) \in B_{\mathcal{F}}^i(\bar{p}, \bar{q}) \text{ and } [P^i(\bar{x}) \times Z^i] \cap B_{\mathcal{F}}^i(\bar{p}, \bar{q}) = \emptyset;$$

$$(b) \sum_{i \in \mathcal{I}} \bar{x}^i = \sum_{i \in \mathcal{I}} e^i \text{ and}$$

$$(c) \sum_{i \in \mathcal{I}} \bar{z}^i = 0.$$

---

<sup>2</sup>For  $x = (x(\xi))_{\xi \in \mathcal{D}}, p = (p(\xi))_{\xi \in \mathcal{D}}$  in  $\mathbb{R}^{\mathbb{L}} = \mathbb{R}^{\mathbb{H} \times \mathbb{D}}$  (with  $x(\xi), p(\xi)$  in  $\mathbb{R}^{\mathbb{H}}$ ) we let  $p \square x = (p(\xi) \bullet_{\mathbb{H}} x(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^{\mathbb{D}}$ .

## Chapter 3

### Restricted Participation and Arbitrage

In a two date (one period) model, Hens *et al.* [47] have shown that, generically, when all investors are unable to participate in all financial markets simultaneously, there will be some arbitrage opportunities that are not exploited. We explore the relevance of this result in a  $T$ -period model with long-lived assets.

Shleifer and Vishny [60] point out that in practice, arbitrage (trading based on the knowledge that the price of an asset is different from its fundamental value) is cheaper for mispriced short term assets than for mispriced long term assets. Thus arbitrageurs would profit more by dealing in mispriced assets for which the mispricing is eliminated in the near future. However they also point out that with perfect capital markets arbitrageurs would not care about the time it takes for a mispriced security to reach its fundamental value. This is because with no transactions costs, absence of short sales, and without the possibility of funds being tied up, the arbitrageur can sell off his risk at no cost.

We consider a general financial model with the possibility of retrading all assets at every period and every contingency. In this model short sales constraints and transaction costs do not exist. Agents would not have costs



associated with tied up funds and thus not care about the time it takes for the mispricing to be eliminated. However as in Hens *et al.* agents are restricted in the different types of assets they can trade in. This could be for reasons such as a bound on information processing capabilities of the agents.

Most results in the two date model carry through in a straightforward manner to the  $T$ -period models with short-lived assets. However this intuition is not immediately clear for the above mentioned result in Hens *et al.*. The natural extension of their result would be that, generically, if at all nodes, all agents were not able to participate in all asset markets simultaneously, then there may be some arbitrage opportunities that are left unexploited. The precise statement is given in Proposition 3.2.3. However, it is possible that at some nodes some agents are able to participate in all markets, and still there may exist some arbitrage opportunities that are not exploited. Example 3.2.1 illustrates this point. With long-lived assets, when re-trading is allowed we can consider a re-trading extension similar in spirit to that in Angeloni and Cornet [1], which helps us to obtain the more general result with  $T$ -periods and possibly long-lived assets.

Section 3.1 describes the  $T$ -period model, the financial structure with long-lived assets along with its re-trading extension and the investor's problem. Section 3.2 characterizes the no-arbitrage conditions with restricted participation and states the central theorem of this paper. We also explain how our result is more general than the natural extension of Hens *et al.* result with short-lived assets. We conclude with the proof of the main theorem.

### 3.1 The model: Special case

The model in this chapter is a special case of the model in Chapter 2. The flow of time and uncertainty is exactly as described in Chapter 2. The differences are in the specification of the exchange economy and the financial structure. Some of the notations will be repeated only to ensure that the set up of the model flows evenly.

#### 3.1.1 The financial structure

There is a finite set  $\mathcal{J} = \{1, \dots, \mathbb{J}\}$  of nominal assets available to transfer income across the different nodes. Each asset  $j \in \mathcal{J}$  is issued at some unique non-terminal node  $\xi \in \mathcal{D}^-$  denoted  $\xi(j)$ . Each asset yields returns only in the nodes succeeding its node of issue. If an asset yields payoffs only in the nodes immediately succeeding the node of issue, it is called a short-lived asset. Otherwise it is called a long lived asset. The payoffs are described by the  $\mathbb{D} \times \mathbb{J}$  matrix  $V$  with entries  $v(\xi, j)$ , representing the payoff of asset  $j$  at node  $\xi$ . Note that the payoffs at nodes preceding the node of issue will be zero. The financial structure is then denoted by  $\mathcal{F} = (\mathcal{J}, (\xi(j))_{j \in \mathbb{J}}, V)$ .

**The re-trading extension:** Given a financial structure

$$\mathcal{F} = (\mathcal{J}, (\xi(j))_{j \in \mathbb{J}}, V)$$

assets can also be re-traded at any node succeeding the node of issue. For any node  $\xi \in \mathcal{D}^-$  we let  $\mathcal{J}(\xi)$  denote the set of assets actively traded at node  $\xi$  and  $\mathbb{J}(\xi)$  its cardinality. This could include assets issued for the first time at this node or assets issued prior to this node and re-traded at this node. Let  $\hat{\mathcal{J}} = \bigcup_{\xi \in \mathcal{D}^-} \mathcal{J}(\xi)$ . We will be considering subsets of assets,  $J \subset \hat{\mathcal{J}}$ , which

can also be written as  $J = \bigcup_{\xi \in \mathcal{D}^-} J(\xi)$  where  $J(\xi) \subset \mathcal{J}(\xi)$ . Let  $\hat{\mathbb{J}}$  denote the cardinality of  $\hat{\mathcal{J}}$ . We assume  $\hat{\mathbb{J}} \leq \#\mathcal{D}^+$ , to allow for possibly incomplete markets.

Suppose asset  $j$  issued at node  $\xi(j)$  is re-traded at some node  $\xi' > \xi(j)$ . We can treat this as a new asset  $\hat{j} = (j, \xi')$  with the issue node  $\xi'$ , i.e.  $\xi(\hat{j}) = \xi'$ . The payoffs from this asset is given by:

$$\begin{aligned} \hat{v}(\xi, \hat{j}) &:= \hat{v}(\xi, (j, \xi')) = v(\xi, j), \text{ if } \xi > \xi' \\ &= 0 \text{ otherwise} \end{aligned}$$

The payoff matrix for this re-trading extension is now given by the  $\mathbb{D} \times \hat{\mathbb{J}}$  matrix  $\hat{V}$ . Thus, given a financial structure  $\mathcal{F} = (\mathcal{J}, (\xi(j))_{j \in \mathbb{J}}, V)$  we can obtain the re-trading extension  $\hat{\mathcal{F}} = (\hat{\mathcal{J}}, (\xi(j))_{j \in \hat{\mathbb{J}}}, \hat{V})$ .

The re-trading extension outlined in Angeloni and Cornet [1], was obtained by considering the re-trading of every asset in all the nodes. Hence  $\hat{\mathbb{J}} = \mathbb{D}\mathbb{J}$  and  $\hat{V}$  was a  $\mathbb{D} \times \mathbb{D}\mathbb{J}$  matrix. However, the exposition here is easier if we let  $\hat{\mathbb{J}}$  contain only the assets that are actively traded.

The asset prices in this re-trading extension are then given by  $q = (q(\xi))_{\xi \in \mathbb{D}^-} = ((q_j)_{j \in \mathbb{J}(\xi)})_{\xi \in \mathbb{D}^-} \in \mathbb{R}^{\hat{\mathbb{J}}}$ . Let  $\hat{\mathcal{V}}$  denote the set of all  $\mathbb{D} \times \hat{\mathbb{J}}$  matrices. Given  $\hat{V} \in \hat{\mathcal{V}}$  and  $q \in \mathbb{R}^{\hat{\mathbb{J}}}$  we can define the  $\mathbb{D} \times \hat{\mathbb{J}}$  total payoff matrix  $W(\hat{V}, q)$  with the following entries:

$$w(\hat{V}, q)(\xi, j) := \hat{v}(\xi, j) - \delta_{\xi, \xi(j)} q_j,$$

where  $\delta_{\xi, \xi'} = 1$  if  $\xi = \xi'$  and  $\delta_{\xi, \xi'} = 0$  otherwise.

Let  $z \in \mathbb{R}^{\hat{\mathbb{J}}}$ . Then for each element  $z_j$ , we have  $j \in \mathcal{J}(\xi)$  for some  $\xi \in \mathcal{D}^-$  and we can interpret,  $z_j > 0$  [resp.  $z_j < 0$ ], as the quantity of asset  $j \in \hat{\mathcal{J}}$  bought [resp. sold] at the trading node  $\xi$ . Thus a vector  $z = (z_j)_{j \in \hat{\mathcal{J}}} \in \mathbb{R}^{\hat{\mathbb{J}}}$

is called a *portfolio*. So, for a given portfolio  $z \in \mathbb{R}^{\mathbb{J}}$  (and given asset price vector  $q$ ) the full flow of payoffs in this re-trading extension is  $W(\hat{V}, q)z$  and the (full) financial payoff at node  $\xi$  is

$$[W(\hat{V}, q)z](\xi) := W(\hat{V}, q)(\xi) \cdot z = \sum_{j \in \hat{\mathcal{J}}} \hat{v}(\xi, j) z_j - \sum_{j \in \hat{\mathcal{J}}} \delta_{\xi, \mathcal{J}(j)} q_j z_j$$

We will occasionally need to consider the payoffs from a portfolio  $z$  as the following sum:

$$\hat{V}z = \sum_{\xi \in \mathcal{D}^-} \hat{V}(\mathcal{J}(\xi))z(\xi)$$

Where  $\hat{V}(\mathcal{J}(\xi))$  is the  $\mathbb{D} \times \mathbb{J}(\xi)$  sub-matrix of  $\hat{V}$  with the columns corresponding to  $j \in \mathcal{J}(\xi)$ .

### 3.1.2 The investor's problem

Each investor  $i \in \mathcal{I}$  is endowed with an initial income stream  $e^i \in \mathbb{R}_+^{\mathbb{D}}$  and a strictly increasing utility function  $u^i : \mathbb{R}_+^{\mathbb{D}} \rightarrow \mathbb{R}$  over the set of possible income streams. Note that here the spot consumption choice is one dimensional, in the framework of the basic model in Chapter 2,  $\mathbb{H} = 1$ , and thus  $\mathbb{L} = \mathbb{D}$ . The spot price of this good is normalized to equal 1. Investors are thus only interested in the incomes they can obtain in each spot. Each investor has the possibility of participating in the asset markets to transfer income across nodes. However we assume that each investor has limited ability in making such transfers. This limitation is imposed on the set of assets that the investor is able to trade in. Thus each investor  $i$  has access only to a subset of assets,  $\mathcal{J}_i \subset \hat{\mathcal{J}}$ . The portfolio set of agent  $i$  is then a subspace of  $\mathbb{R}^{\hat{\mathcal{J}}}$ .

Investor  $i$ 's optimization problem is then given as:

$$\max_{(x,z) \in \mathcal{B}(\hat{V}, q)} U^i(x)$$

where

$$\mathcal{B}(\hat{V}, q) = \{(x, z) \in \mathbb{R}_+^{\mathbb{D}} \times \mathbb{R}^{\hat{\mathbb{J}}} \mid x - e^i \leq W(\hat{V}, q)z, \forall j \notin \mathcal{J}_i, z_j = 0\}$$

### 3.2 No-arbitrage condition with restricted participation

**Definition 3.2.1**  $NAC^J$ : Given  $\hat{V} \in \hat{\mathcal{V}}, q \in \mathbb{R}^{\hat{\mathbb{J}}}$  and  $J \subset \hat{\mathcal{J}}$ , we say there are no arbitrage opportunities in the  $J$  markets if the following holds:

$$\nexists z \in \mathbb{R}^{\hat{\mathbb{J}}}, \text{ with } z_j = 0, \forall j \notin J, \text{ such that } W(\hat{V}, q)z > 0$$

It is a standard result that  $NAC^J$  holds if and only if  $q \in \mathcal{Q}(\hat{V}, J)$ , where

$$\mathcal{Q}(\hat{V}, J) := \{q \in \mathbb{R}^{\hat{\mathbb{J}}} \mid \exists \lambda \in \mathbb{R}_{++}^{\mathbb{D}} \text{ such that } \forall j \in J, \lambda(\xi(j))q_j = \sum_{\xi' > \xi(j)} \lambda(\xi')\hat{v}(\xi', j)\}.$$

**Remark 3.2.1** The following are evident from this definition:

1. For  $i \in \mathcal{I}$ , the assets available are  $J_i$ . Setting  $J = J_i$  in the above definition we can say that agent  $i$  has no arbitrage opportunities if and only if  $q \in \mathcal{Q}(\hat{V}, J_i)$ .
2. Setting  $J = \hat{\mathcal{J}}$  in the above definition we can say that there are no arbitrage opportunities in all markets if and only if  $q \in \mathcal{Q}(\hat{V}, \hat{\mathcal{J}})$ . Notice that we can write

$$\mathcal{Q}(\hat{V}, \hat{\mathcal{J}}) = \{q \in \mathbb{R}^{\hat{\mathbb{J}}} \mid \exists \lambda \in \mathbb{R}_{++}^{\mathbb{D}} \text{ such that } {}^tW(\hat{V}, q)\lambda = 0\}.$$

**Proposition 3.2.1** *Given  $\hat{V} \in \hat{\mathcal{V}}$  the following are evident:*

1.  $J \subset J' \Rightarrow \mathcal{Q}(\hat{V}, J') \subset \mathcal{Q}(\hat{V}, J)$ .
2.  $\mathcal{Q}(\hat{V}, \hat{\mathcal{J}}) \subset \mathcal{Q}(\hat{V}, \bigcup_{i \in \mathcal{I}} J_i) \subset \bigcap_{i \in \mathcal{I}} \mathcal{Q}(\hat{V}, J_i)$ .
3. *If there exists  $i_0 \in \mathcal{I}$  such that  $J_{i_0} = \hat{\mathcal{J}}$  then  $\mathcal{Q}(\hat{V}, \hat{\mathcal{J}}) = \bigcap_{i \in \mathcal{I}} \mathcal{Q}(\hat{V}, J_i)$ .*

**Proof.**

1. This fact is easy to see from the definition of  $\mathcal{Q}(\hat{V}, J)$ .
2.  $\bigcup_{i \in \mathcal{I}} J_i \subset \hat{\mathcal{J}}$ , thus we have the first inclusion. For all  $i \in \mathcal{I}$ ,  $J_i \subset \bigcup_{i \in \mathcal{I}} J_i$  implies, for all  $i \in \mathcal{I}$ ,  $\mathcal{Q}(\hat{V}, \bigcup_{i \in \mathcal{I}} J_i) \subset \mathcal{Q}(\hat{V}, J_i)$ . Thus  $\mathcal{Q}(\hat{V}, \bigcup_{i \in \mathcal{I}} J_i) \subset \bigcap_{i \in \mathcal{I}} \mathcal{Q}(\hat{V}, J_i)$
3. It is obvious that  $\mathcal{Q}(\hat{V}, \hat{\mathcal{J}}) \subset \bigcap_{i \in \mathcal{I}} \mathcal{Q}(\hat{V}, J_i)$  always. If  $q \in \bigcap_{i \in \mathcal{I}} \mathcal{Q}(\hat{V}, J_i)$  then for all  $i \in \mathcal{I}$ ,  $q \in \mathcal{Q}(\hat{V}, J_i)$ . In particular  $q \in \mathcal{Q}(\hat{V}, J_{i_0}) = \mathcal{Q}(\hat{V}, \hat{\mathcal{J}})$  since  $J_{i_0} = \hat{\mathcal{J}}$ .

□

Given  $\hat{V} \in \hat{\mathcal{V}}$  and  $J \subset \hat{\mathcal{J}}$ , define

$$\mathcal{Z}(\hat{V}, J) := \{z \in \mathbb{R}^{\hat{\mathcal{J}}} \mid z_j = 0, \forall j \notin J, \text{ and } \hat{V}z \geq 0\}$$

**Lemma 3.2.1** *Given  $\hat{V} \in \hat{\mathcal{V}}$  and  $J \subset \hat{\mathcal{J}}$  we have the following<sup>1</sup>:*

1.  $\mathcal{Q}(\hat{V}, J)$  is a convex cone.
2.  $\mathcal{Q}^\oplus(\hat{V}, J) \subset \mathcal{Z}(\hat{V}, J)$

---

<sup>1</sup>Given  $A \subset \mathbb{R}^n$  define the positive polar cone to  $A$  by  $A^\oplus := \{x \in \mathbb{R}^n \mid a \cdot x \geq 0 \forall a \in A\}$ .

**Proof.** We will set up the proof using the following definitions. Let  $\hat{V} \in \hat{\mathcal{V}}$  and  $J \subset \hat{\mathcal{J}}$ .

- No-arbitrage prices at trading nodes ( $NAC(J, \xi)$ ):  $\forall \xi \in \mathcal{D}^-$  define

$$\mathcal{Q}(\hat{V}, J, \xi) = \{q(\xi) \in \mathbb{R}^{\#\mathbb{J}(\xi)} \mid \exists \pi \in \mathbb{R}_{++}^{\#\mathcal{D}^+(\xi)} \text{ such that}$$

$$q_j(\xi) = \sum_{\xi' > \xi} \pi(\xi') \hat{v}(\xi', j), \forall j \in J(\xi)\}$$

- Using Farkas' lemma we can conclude that:

$$\mathcal{Q}^\oplus(\hat{V}, J, \xi) = \{z(\xi) \in \mathbb{R}^{\#\mathcal{J}(\xi)} \mid z_j = 0 \forall j \notin J(\xi), \hat{V}(\mathbb{J}(\xi))z(\xi) \geq 0\}$$

- We can also define the following:

$$\tilde{\mathcal{Q}}(\hat{V}, J, \xi) := \{q \in \mathbb{R}^{\hat{\mathbb{J}}} \mid q(\xi) \in \mathcal{Q}(\hat{V}, J, \xi)\}$$

- Using Farkas' lemma again, we see that

$$\tilde{\mathcal{Q}}^\oplus(\hat{V}, J, \xi) = \{z \in \mathbb{R}^{\#J(\xi)} \mid z_j = 0 \forall j \notin J(\xi), \hat{V}z \geq 0\}$$

1. Note that  $\mathcal{Q}(\hat{V}, J) = \prod_{\xi \in \mathcal{D}^-} \mathcal{Q}(\hat{V}, J, \xi)$  and thus a convex cone. To see this equality one inclusion is straight forward, if  $q$  is a no-arbitrage price then  $q(\xi)$  is a no-arbitrage price for all  $\xi \in \mathcal{D}^-$ . To see the reverse inclusion, let  $q \in \prod_{\xi \in \mathcal{D}^-} \mathcal{Q}(\hat{V}, J, \xi)$ . Then for all  $\xi \in \mathcal{D}^-$ ,  $\exists \pi^\xi \in \mathbb{R}_{++}^{\#\mathcal{D}^+(\xi)}$  such that ( $NAC(J, \xi)$ ) holds. Let  $\lambda(\xi_0) = 1$  and for all  $\xi \in \mathcal{D}^+$  let  $\lambda(\xi) = \lambda(\xi^-) \pi^{\xi^-}(\xi)$ . Then  $q \in \mathcal{Q}(\hat{V}, J)$  with the associated  $\lambda$  defined above.

2. Note that  $\mathcal{Q}(\hat{V}, J) = \prod_{\xi \in \mathcal{D}^-} \mathcal{Q}(\hat{V}, J, \xi) = \bigcap_{\xi \in \mathcal{D}^-} \tilde{\mathcal{Q}}(\hat{V}, J, \xi)$ . So

$$\mathcal{Q}^\oplus(\hat{V}, J) = \left( \bigcap_{\xi \in \mathcal{D}^-} \tilde{\mathcal{Q}}(\hat{V}, J, \xi) \right)^\oplus = \sum_{\xi \in \mathcal{D}^-} \tilde{\mathcal{Q}}^\oplus(\hat{V}, J, \xi).$$

The last equality is obtained by applying corollary 16.4.2 in Rockafellar (1997) [59]. Finally, note that

$$\sum_{\xi \in \mathcal{D}^-} \tilde{\mathcal{Q}}^\oplus(\hat{V}, J, \xi) \subset Z(\hat{V}, J).$$

□

Lets define the set of matrices for which there is some portfolio yielding non-negative payoffs in every future state by,

$$\hat{\mathcal{V}}_+ = \{\hat{V} \in \hat{\mathcal{V}} \mid \exists z \in \mathbb{R}^{\hat{\mathcal{J}}} \setminus \{0\} \text{ such that } \hat{V}z \geq 0\}.$$

The following proposition states that matrices outside of  $\hat{\mathcal{V}}_+$  are hardly interesting.

**Proposition 3.2.2** *It holds that  $\hat{V} \notin \hat{\mathcal{V}}_+ \implies \mathcal{Q}(\hat{V}, \hat{\mathcal{J}}) = \mathbb{R}^{\hat{\mathcal{J}}}$ .*

**Proof.**  $\hat{V} \notin \mathcal{V}_+ \Leftrightarrow \mathcal{Z}(V, \hat{\mathcal{J}}) = \{0\} \Rightarrow cl \mathcal{Q}(\hat{V}, \hat{\mathcal{J}}) = \mathbb{R}^{\hat{\mathcal{J}}}$ .<sup>2</sup> However since  $\mathcal{Q}(\hat{V}, \hat{\mathcal{J}})$  is a convex cone we have:  $\mathbb{R}^{\hat{\mathcal{J}}} = ri(\mathbb{R}^{\hat{\mathcal{J}}}) = ri(cl(\mathcal{Q}(\hat{V}, \hat{\mathcal{J}}))) = ri(\mathcal{Q}(\hat{V}, \hat{\mathcal{J}})) \subset \mathcal{Q}(\hat{V}, \hat{\mathcal{J}}) \subset cl(\mathcal{Q}(\hat{V}, \hat{\mathcal{J}})) \subset \mathbb{R}^{\hat{\mathcal{J}}}$ . Thus we have  $\mathbb{R}^{\hat{\mathcal{J}}} = \mathcal{Q}(\hat{V}, \hat{\mathcal{J}})$ .

□

### 3.2.1 The main theorem

The following theorem is the main result of this paper. Note that Hens *et al.* [47] implicitly assume that  $\bigcup_{i \in \mathcal{I}} J_i = \hat{\mathcal{J}}$ . Which is reasonable because, if none of the agents are able to trade in a certain asset at any node, then we can eliminate that asset from the list and call the remaining list of assets as  $\hat{\mathcal{J}}$ . In what follows we will also assume that  $\bigcup_{i \in \mathcal{I}} J_i = \hat{\mathcal{J}}$ .

---

<sup>2</sup>Given  $A \subset \mathbb{R}^n$ ,  $cl(A)$  (resp.  $ri(A)$ ) is the closure of (resp. relative interior) of  $A$ .



**Theorem 3.2.1** *Suppose for all investors  $i \in \mathcal{I}$  it holds that  $J_i \neq \hat{\mathcal{J}}$ . Then there exists an open subset  $\hat{\mathcal{V}}_+^*$  of the set  $\hat{\mathcal{V}}_+$  with  $\hat{\mathcal{V}}_+ \setminus \hat{\mathcal{V}}_+^*$  having Lebesgue measure zero such that  $\mathcal{Q}(\hat{V}, \hat{\mathcal{J}}) \neq \bigcap_{i \in \mathcal{I}} \mathcal{Q}(\hat{V}, J_i)$  for all  $\hat{V} \in \hat{\mathcal{V}}_+^*$ .*

Before entering the proof of this theorem we make the following observations. In the case of only short-lived assets, the result of Hens *et al.* can be extended in a natural way to prove the following proposition, the proof of which is given in the appendix.

**Proposition 3.2.3** *Suppose all assets are short-lived and for all nodes  $\xi \in \mathcal{D}^-$  and for all investors  $i \in \mathcal{I}$  it holds that  $J_i(\xi) \neq \mathcal{J}(\xi)$ . Then there exists an open subset  $\hat{\mathcal{V}}_+^*$  of the set  $\hat{\mathcal{V}}_+$  with  $\hat{\mathcal{V}}_+ \setminus \hat{\mathcal{V}}_+^*$  having Lebesgue measure zero such that  $\mathcal{Q}(\hat{V}, \hat{\mathcal{J}}) \neq \bigcap_{i \in \mathcal{I}} \mathcal{Q}(\hat{V}, J_i)$  for all  $\hat{V} \in \hat{\mathcal{V}}_+^*$ .*

**Remark 3.2.2** *Notice that Theorem 3.2.1 is more general than the natural extension of the Hens *et al.* result in Proposition 3.2.3. In the  $T > 1$  case it is possible that  $J_i \neq \hat{\mathcal{J}}$  and for all  $\xi \in \mathcal{D}^-$  there exists  $i \in \mathcal{I}$  such that  $J_i(\xi) = \mathcal{J}(\xi)$ . See the example 3.2.1.*

**Example 3.2.1** *Suppose all assets are short-lived. Let  $\mathcal{I} = \{1, 2, 3\}$  and  $T = 2$ . Let the date event tree be given by  $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6\}$ ,  $\xi_0^+ = \{\xi_1, \xi_2\}$ ,  $\xi_1^+ = \{\xi_3, \xi_4\}$ ,  $\xi_2^+ = \{\xi_5, \xi_6\}$ .*

*Let the set of assets be  $\mathcal{J} = \{1, 2, 3, 4, 5\}$  with  $\mathcal{J}(\xi_0) = \{1, 2\}$ ,  $\mathcal{J}(\xi_1) = \{3, 4\}$ ,  $\mathcal{J}(\xi_2) = \{5\}$  and the participation structure be given by  $J_1 = \{2, 3, 4, 5\}$ ,  $J_2 = \{1, 2, 4, 5\}$  and  $J_3 = \{1, 3\}$ . Then  $\forall \xi \in \mathcal{D}^-, \exists i \in \mathcal{I}$  such that  $J_i(\xi) = \mathcal{J}(\xi)$  however  $\forall i \in \mathcal{I}, J_i \neq \mathcal{J}$ .*

Theorem 3.2.1 is more general than the natural extension of the Hens *et al.* result and includes the case of long-lived assets.

### 3.2.2 Proof of main theorem

The set up for the proof of Theorem 3.2.1 is done through the following lemmas.

For  $z \neq 0$ , denote  $\langle z \rangle := \{\alpha z \mid \alpha \geq 0\}$ , the half-line spanned by the vector  $z$ . Given  $\hat{V} \in \hat{\mathcal{V}}$  denote by  $T(\hat{V}, \hat{\mathcal{J}})$  the set of  $z \in \mathcal{Z}(\hat{V}, \hat{\mathcal{J}})$  such that  $\langle z \rangle$  is an extreme ray of  $\mathcal{Z}(\hat{V}, \hat{\mathcal{J}})$ <sup>3</sup>.

**Lemma 3.2.2** *Let  $\hat{V} \in \hat{\mathcal{V}}_+$  with  $\text{rank } \hat{V} = \hat{\mathbb{J}}$ . Then we have the following:*

1.  $\mathcal{Z}(\hat{V}, \hat{\mathcal{J}}) = \text{cc } \{z \in T(\hat{V}, \hat{\mathcal{J}}) \mid \|z\| = 1\}$ <sup>4</sup>.
2. Let  $z \in \mathcal{Z}(\hat{V}, \hat{\mathcal{J}}) \setminus \{0\}$ . Then  $z \in T(\hat{V}, \hat{\mathcal{J}}) \iff \text{rank } \{\hat{V}(\xi) \mid \xi \in \mathbb{D}(z)\} = \hat{\mathbb{J}} - 1$ , where  $\mathbb{D}(z) := \{\xi \in \mathbb{D} \mid \hat{V}(\xi) \cdot z = 0\}$ .

**Proof.**

1. Since  $\hat{V} \in \hat{\mathcal{V}}_+$  and  $\text{rank } (\hat{V}) = \hat{\mathbb{J}}, \exists z \in \mathcal{Z}(\hat{V}, \hat{\mathcal{J}}) \setminus \{0\}$  and  $\mathcal{Z}(\hat{V}, \hat{\mathcal{J}})$  contains no lines. By corollary 18.5.2 Rockafellar (1997)[59] we have the result.
2. The proof follows easily from Proposition 3.3.2 Florenzano *et al.* [30].

□

**Lemma 3.2.3** *Let  $\hat{V} \in \hat{\mathcal{V}}_+$  with  $\text{rank } (\hat{V}) = \hat{\mathbb{J}}$ . Then the following are equivalent:*

1.  $\mathcal{Q}(\hat{V}, \hat{\mathcal{J}}) = \bigcap_{i \in \mathcal{I}} \mathcal{Q}(\hat{V}, J_i)$

---

<sup>3</sup>Given  $C \subset \mathbb{R}^n$  a convex cone, and  $x \in C \setminus \{0\}$ .  $\langle x \rangle$  is an extreme ray of  $C$  if and only if every line segment with a relative interior point in  $\langle x \rangle$  has both end points in  $\langle x \rangle$ .

<sup>4</sup>Given  $A \subset \mathbb{R}^n$ ,  $\text{cc } (A)$  is the convex cone generated by the vectors in  $A$ .

$$2. \mathcal{Z}(\hat{V}, \hat{\mathcal{J}}) = \sum_{i \in \mathcal{I}} \mathcal{Z}(\hat{V}, J_i)$$

$$3. T(\hat{V}, \hat{\mathcal{J}}) \subset \bigcup_{i \in \mathcal{I}} \mathcal{Z}(\hat{V}, J_i)$$

**Proof.** (1)  $\Leftrightarrow$  (2) :

$$\mathcal{Z}(\hat{V}, \hat{\mathcal{J}}) = \left( \bigcap_{i \in \mathcal{I}} \mathcal{Q}(\hat{V}, J_i) \right)^\oplus = \left( \bigcap_{i \in \mathcal{I}} \mathcal{Q}^{\oplus \oplus}(\hat{V}, J_i) \right)^\oplus = \left( \sum_{i \in \mathcal{I}} \mathcal{Z}(\hat{V}, J_i) \right)^{\oplus \oplus} = \sum_{i \in \mathcal{I}} \mathcal{Z}(\hat{V}, J_i)$$

Using (1) and Lemma 3.2.1 we have the first equality. The third equality is due to corollary 16.4.2 Rockafellar 1997 [59].

(2)  $\Rightarrow$  (3) : Let  $z \in T(\hat{V}, \hat{\mathcal{J}})$ , then  $z \neq 0$  and  $z \in \mathcal{Z}(\hat{V}, \hat{\mathcal{J}})$ . By (2), for all  $i \in \mathcal{I}$  there exists  $z^i \in \mathcal{Z}(\hat{V}, J_i)$  such that  $z = \sum_{i \in \mathcal{I}} z^i$ . Since  $z \neq 0$ ,  $\exists i_0 \in \mathcal{I}$  such that  $z^{i_0} \neq 0$  in the above summation. Notice that the segment  $[2z^{i_0}, 2 \sum_{i \in \mathcal{I} \setminus \{i_0\}} z^i]$  contains  $z$  in its relative interior. Since  $\langle z \rangle$  is an extreme ray of  $\mathcal{Z}(\hat{V}, \hat{\mathcal{J}})$ , both end points of the segment belong to  $\langle z \rangle$ . So there exists  $t > 0$  such that  $2z^{i_0} = tz$ . Thus  $z \in \mathcal{Z}(\hat{V}, J_{i_0}) \subset \bigcup_{i \in \mathcal{I}} \mathcal{Z}(\hat{V}, J_i)$ .

(3)  $\Rightarrow$  (2) :  $\sum_{i \in \mathcal{I}} \mathcal{Z}(\hat{V}, J_i) \subset \mathcal{Z}(\hat{V}, \hat{\mathcal{J}})$  is obvious.  $T(\hat{V}, \hat{\mathcal{J}})$  generates  $\mathcal{Z}(\hat{V}, \hat{\mathcal{J}})$  and  $\bigcup_{i \in \mathcal{I}} \mathcal{Z}(\hat{V}, J_i)$  generates  $\sum_{i \in \mathcal{I}} \mathcal{Z}(\hat{V}, J_i)$ . Thus  $\mathcal{Z}(\hat{V}, \hat{\mathcal{J}}) \subset \sum_{i \in \mathcal{I}} \mathcal{Z}(\hat{V}, J_i)$ .

□

**Lemma 3.2.4** *Suppose that  $J_i \neq \hat{\mathcal{J}}$  for all  $i \in \mathcal{I}$ . Then there exists an open subset  $\hat{\mathcal{V}}'$  of  $\hat{\mathcal{V}}$  with  $\hat{\mathcal{V}} \setminus \hat{\mathcal{V}}'$  having Lebesgue measure zero such that for all  $\hat{V} \in \hat{\mathcal{V}}' \cap \hat{\mathcal{V}}_+$ ,*  

$$T(\hat{V}, \hat{\mathcal{J}}) \cap \left( \bigcup_{i \in \mathcal{I}} \mathcal{Z}(\hat{V}, J_i) \right) = \emptyset.$$

**Proof.** For every  $j \in \hat{\mathcal{J}}$  and for every subset  $M \subset \mathcal{D}$  with cardinality  $\mathbb{J} - 1$ ,

define the function  $F_{jM} : \hat{\mathcal{V}} \times \mathbb{R}^{\hat{\mathbb{J}}} \rightarrow \mathbb{R}^{\hat{\mathbb{J}}+1}$  as follows,

$$F_{jM}(\hat{V}, z) = \begin{bmatrix} \hat{V}^\xi \cdot z, \xi \in M \\ z \cdot z - 1 \\ z_j \end{bmatrix}.$$

Define the sets  $\hat{\mathcal{V}}_{jM}$  as

$$\hat{\mathcal{V}}_{jM} = \left\{ \hat{V} \in \hat{\mathcal{V}} \mid \text{there is no } z \in \mathbb{R}^{\hat{\mathbb{J}}} \text{ such that } F_{jM}(\hat{V}, z) = 0 \right\}.$$

To see that  $\hat{\mathcal{V}}_{jM}$  is open, let  $\hat{V}_{(n)}$  be the sequence of matrices in  $\hat{\mathcal{V}} \setminus \hat{\mathcal{V}}_{jM}$  converging to some  $\hat{V} \in \hat{\mathcal{V}}$ . Then there exists a sequence  $z_{(n)}$  in  $\mathbb{R}^{\hat{\mathbb{J}}}$  such that  $F_{jM}(\hat{V}_{(n)}, z_{(n)}) = 0$  for all  $n$ . Since the sequence  $z_{(n)}$  is bounded, it has a convergent subsequence converging to some  $z \in \mathbb{R}^{\hat{\mathbb{J}}}$ . Hence,  $F_{jM}(\hat{V}, z) = 0$ , and the matrix  $\hat{V}$  belongs to the complement of the set  $\hat{\mathcal{V}}_{jM}$ .

It is clear that for all  $(\hat{V}, z) \in F_{jM}^{-1}(0)$  the matrix of the partial derivatives has full row rank. That is,  $F_{jM}$  is transversal to zero. The Transversality Theorem implies that the complement of the set  $\hat{\mathcal{V}}_{jM}$  has Lebesgue measure zero.

Define  $\hat{\mathcal{V}}'$  as the set of matrices with rank  $\hat{\mathbb{J}}$  in the intersection of all sets  $\hat{\mathcal{V}}_{jM}$ . Then  $\hat{\mathcal{V}}'$  is open and its complement has Lebesgue measure zero.

Let  $\hat{V} \in \hat{\mathcal{V}}' \cap \hat{\mathcal{V}}_+$  and  $z \in T(\hat{V}, \hat{\mathcal{J}})$ . Suppose that  $z \in \mathcal{Z}(\hat{V}, J_i)$  for some  $i \in \mathcal{I}$ . Then  $z_j = 0$  for every  $j \in \hat{\mathcal{J}} \setminus J_i$ . Lemma 3.2.2 implies that there is a subset  $M$  of the set  $\mathbb{D}$  with cardinality  $\hat{\mathbb{J}} - 1$  such that  $\hat{V}(\xi) \cdot z = 0$  for all  $\xi \in M$ . Therefore,  $F_{jM}(\hat{V}, z) = 0$  for every  $j \in \hat{\mathcal{J}} \setminus J_i$ , a contradiction to  $\hat{V} \in \hat{\mathcal{V}}'$ . Thus, we have proved that  $T(\hat{V}, \hat{\mathcal{J}}) \cap \mathcal{Z}(\hat{V}, J_i) = \emptyset$  for all  $i \in \mathcal{I}$  and  $\hat{V} \in \hat{\mathcal{V}}' \cap \hat{\mathcal{V}}_+$ .  $\square$

**Proof. of Theorem 3.2.1**

Define  $\hat{\mathcal{V}}_+^*$  to be the set of matrices with rank  $\hat{\mathbb{J}}$  in  $\hat{\mathcal{V}}_+ \cap \hat{\mathcal{V}}'$ . Since  $\hat{\mathcal{V}}'$  and the set of matrices with rank  $\hat{\mathbb{J}}$  are open in  $\hat{\mathcal{V}}_+$ ,  $\hat{\mathcal{V}}_+^*$  is open in  $\hat{\mathcal{V}}_+$ . Also since  $\hat{\mathcal{V}}'$  and the set of matrices with rank  $\hat{\mathbb{J}}$  have full Lebesgue measure, the set  $\hat{\mathcal{V}}_+ \setminus \hat{\mathcal{V}}_+^*$  has Lebesgue measure zero. By Lemma 3.2.4, for all  $\hat{V} \in \hat{\mathcal{V}}_+^*$ ,  $T(\hat{V}, \hat{\mathcal{J}})$  is nonempty, however for all  $i \in \mathcal{I}$ ,  $T(\hat{V}, \hat{\mathcal{J}}) \cap \mathcal{Z}(\hat{V}, J_i) = \emptyset$ . Thus by Lemma 3.2.3 we have  $\bigcap_{i \in \mathcal{I}} \mathcal{Q}(\hat{V}, \hat{\mathcal{J}}) \neq \mathcal{Q}(\hat{V}, J_i)$ .  $\square$

### 3.3 Appendix to Chapter 3

With only short-lived assets, we have  $\mathcal{J} = \hat{\mathcal{J}}$  and thus  $V = \hat{V}$ . Before we begin the proof of Proposition 3.2.3, we introduce the following notations for each  $\xi \in \mathbb{D}^-$ :

1.  $\mathcal{V}(\xi)$  is the set of  $\#\xi^+ \times \#\mathcal{J}(\xi)$  matrices
2.  $z(\xi)$  and  $q(\xi)$  are vectors in  $\mathbb{R}^{\#\mathcal{J}(\xi)}$
3.  $\mathcal{V}_+(\xi) := \{V(\xi) \mid \exists z(\xi) \neq 0 \text{ such that } V(\xi)z(\xi) \geq 0\}$
4. Let  $\mu^\xi$  (*resp.*  $\mu$ ) represent the Lebesgue measure on  $\mathcal{V}(\xi)$  (*resp.*  $\mathcal{V}$ )
5.  $\mathcal{Q}(V(\xi), J(\xi)) := \{q(\xi) \mid \exists \lambda \in \mathbb{R}_{++}^{\#\xi^+} \text{ such that } \forall j \in J(\xi), q_j = \sum_{\sigma \in \xi^+} \lambda(\sigma) v(\sigma, j)\}$   
where  $J(\xi) \subset \mathcal{J}(\xi)$ .

We now make the following observations:<sup>5</sup>

1.  $\mathcal{V} = \bigoplus_{\xi \in \mathcal{D}^-} \mathcal{V}(\xi) = \{V = \bigoplus_{\xi \in \mathcal{D}^-} V(\xi) \mid V(\xi) \in \mathcal{V}(\xi)\}$

---

<sup>5</sup>Given matrices  $A_n$  of dimensions  $l_n \times m_n$  for  $n = 1, \dots, N$ , the direct sum matrix is denoted  $\bigoplus_{n \in \{1, \dots, N\}} A_n = A_1 \oplus \dots \oplus A_N$  which is of dimension  $\sum_{n=1}^N l_n \times \sum_{n=1}^N m_n$ .

$$2. \mathcal{V}_+ = \bigoplus_{\xi \in \mathcal{D}^-} \mathcal{V}_+(\xi) = \{V = \bigoplus_{\xi \in \mathcal{D}^-} V(\xi) \mid V(\xi) \in \mathcal{V}_+(\xi)\}$$

$$3. \text{ Given } \mathcal{V}' \subset \mathcal{V}, \mu \text{ is now naturally defined by } \mu(\mathcal{V}') = \prod_{\xi \in \mathcal{D}^-} \mu^\xi(\mathcal{V}'(\xi))$$

### Proof of Proposition 3.2.3

If for all  $\xi \in \mathcal{D}^-$  and for all  $i \in \mathcal{I}$ ,  $J_i(\xi) \neq \mathcal{J}(\xi)$  then for all  $\xi \in \mathcal{D}^-$  there exists an open subset  $\mathcal{V}_+^*(\xi) \subset \mathcal{V}_+(\xi)$  with full Lebesgue measure such that  $\mathcal{Q}(V(\xi), \mathcal{J}(\xi)) \neq \bigcap_{i \in \mathcal{I}} \mathcal{Q}(V(\xi), J_i(\xi))$  for all  $V(\xi) \in \mathcal{V}_+^*(\xi)$ . This is due to the Hens *et al.* result.

Notice that  $\mathcal{V}_+^* = \bigoplus_{\xi \in \mathcal{D}^-} \mathcal{V}_+^*(\xi)$  is an open subset of  $\mathcal{V}_+$ . Let  $V = \bigoplus_{\xi \in \mathcal{D}^-} V(\xi) \in \mathcal{V}_+ \setminus \mathcal{V}_+^*$ . Since for all  $\xi \in \mathcal{D}^-$ ,  $\mathcal{V}_+^*(\xi)$  is open, there exists a sequence  $V_n(\xi)$  in  $\mathcal{V}_+(\xi) \setminus \mathcal{V}_+^*(\xi)$  converging to  $V(\xi) \in \mathcal{V}_+(\xi) \setminus \mathcal{V}_+^*(\xi)$ . Thus there exists a sequence  $V_n$  in  $\mathcal{V}_+ \setminus \mathcal{V}_+^*$  converging to  $V \in \mathcal{V}_+ \setminus \mathcal{V}_+^*$ . Thus  $\mathcal{V}_+^*$  is open.

Also, notice that since for each  $\xi \in \mathcal{D}^-$ ,  $\mathcal{V}_+(\xi) \setminus \mathcal{V}_+^*(\xi)$  has Lebesgue measure zero. Thus the set  $\mathcal{V}_+ \setminus \mathcal{V}_+^*$  has Lebesgue measure zero.

Since for all  $\xi \in \mathcal{D}^-$  and for all  $V(\xi) \in \mathcal{V}_+^*(\xi)$ ,  $\mathcal{Q}(V(\xi), \mathcal{J}(\xi)) \neq \bigcap_{i \in \mathcal{I}} \mathcal{Q}(V(\xi), J_i(\xi))$  we have for all  $V \in \mathcal{V}_+^*$ ,  $\mathcal{Q}(V, \mathcal{J}) \neq \bigcap_{i \in \mathcal{I}} \mathcal{Q}(V, J_i)$ .  $\square$

## Chapter 4

### Equilibria with Restricted Participation

Investors facing restrictions on the portfolios that they can trade, is more of a norm than an exception. We consider a model in which investors' portfolio sets are constrained. As in Balasko, Cass and Siconolfi [6] these constraints are exogenously given (probably arising due to some institutional reasons). Moreover, we consider very general restrictions on portfolio sets as in Siconolfi [61], where each agents portfolios set is assumed to be convex and containing zero.

This paper primarily examines the existence of a financial equilibrium in a multiperiod model when investors face such general portfolio restrictions. In two date (one period) models without restrictions on portfolio sets, the existence issue has been extensively studied. Cass ([13]) and Werner ([65], [67]) showed existence with nominal assets. Duffie and Shafer ([26]) showed a generic existence result with real assets. This second approach has been extensively used. Magill and Shafer [52] provide a good survey of financial markets equilibria and contingent markets equilibria. Another approach to prove existence in a differentiable economy is to show existence in a numeraire asset economy and infer the existence in the nominal asset economy (See Villanacci

et al. Villanacci et al and Magill and Quinzii [53]).

Multiperiod models are better equipped to capture the evolution of time and uncertainty and are a necessary step before studying infinite horizon models. Following Debreu's [20] pioneering model we consider an event-tree to represent the evolution of time and uncertainty. Magill and Quinzii ([53]) and Angeloni and Cornet ([1]) are great references for the treatment of multiperiod financial models. Each node in the event tree represents a date event. Given information on asset prices and spot prices at all date events, consumers will choose a consumption and a portfolio of assets (assumed to be constrained here), such that the node specific value of the consumption does not exceed the node specific value of their endowments and the net returns from the portfolio.

In the absence of such portfolio restrictions the notion of absence of arbitrage is clear - if there is no portfolio that yields nonnegative net returns in all nodes and strictly positive returns in some node. However in the case where all agents face restrictions in their asset market participation, the notion of arbitrage and its absence at the individual level may differ from that at the aggregate level. Angeloni and Cornet [1] make this distinction. Given asset prices a portfolio for an agent does not offer arbitrage opportunities if she cannot find another portfolio within her constrained portfolio set that yields a higher net payoff returns in all nodes and strictly positive payoff in some node. On the other hand, there are no arbitrage opportunities in the aggregate, if there is no portfolio in the set of pooled portfolio sets of all agents that yields nonnegative net payoffs in all nodes and strictly positive payoff in some node.

At an equilibrium there must be no arbitrage at the individual level. A



natural question then is will any asset price at which there is no arbitrage be an equilibrium asset price. The objective of this paper is to explore this characterization under general portfolio constraints.

In the absence of portfolio constraints, Cass ([13]), Duffie ([24]) and Florenzano and Gourdel ([29]) show this characterization of equilibrium and arbitrage free asset prices. In the presence of such constraints, the approach initiated by Cass ([13]), where one agent has an unconstrained portfolio set, facilitates the existence proof. This approach has been extensively used to show existence, Magill and Shafer ([52]), Florenzano and Gourdel ([29]), Magill and Quinzii ([53]), Angeloni and Cornet ([1]) among others.

This approach of Cass ([13]), breaks the symmetry of the problem and hence it is not possible to give a symmetric proof of existence (symmetric with respect to the agents' problem). More recently in a working paper, with such general portfolio restrictions, Da-Rocha and Triki ([54]) have been able to show the characterization between equilibrium and arbitrage free asset prices without the use of the Cass approach. Hahn and Won ([41]), are also able to show this albeit with monotonic preferences and more involved notion of "Projective" arbitrage. However the notion of arbitrage and its absence in these two papers differ from each other and from the notion considered in this paper, that of Angeloni and Cornet ([1]).

In this paper we explore this characterization issue by showing that any market arbitrage free asset price can be supported as an equilibrium asset price. The approach here is similar to that in Da-Rocha and Triki ([54]), however the notion of absence of arbitrage and the condition on agents' set of income transfers - *transfer space condition*, is weaker than those in Da-Rocha

and Triki ([54]). The transfer space condition we require in this paper is, for any payoff that can be obtained through the pooled portfolio set there is some agent who can obtain a fraction of this payoff through her portfolio set and there is some agent who can obtain a negative of a fraction of this payoff through her portfolio set. We can interpret this as, for any payoff that is possible for all agents pooled together, an asset yielding a fraction of that payoff can be short sold or bought by some agent.

In the Cass approach, the unconstrained agent behaves like in an Arrow-Debreu economy and is able to accommodate the equilibrium excess demand for assets. The attainable consumption and asset allocation are then bounded. We observe that this assumption can be improved upon considerably. If the set of attainable income transfers is bounded then we can guarantee the existence of a weak-equilibrium, which differs from an equilibrium only in the requirement that instead of asset market clearing, there is accounts clearing in the asset markets. The notion of weak equilibrium is useful for instance when redundant assets exist.

Section 4.1 describes this notions of absence of arbitrage at the individual and aggregate level and their relation to equilibrium. Section 4.2, states the main result and discusses the notion of a weak-equilibrium and its existence. Section 4.3.1 discusses the various notions of absence of arbitrage and the compatibility conditions on the space of income transfers needed to guarantee the existence of an equilibrium. The previous results in this area are listed as corollaries. Section 4.5 gives a detailed proof of the central result in this chapter.

## 4.1 Arbitrage and equilibrium

This chapter relies on the general framework given by the model in Chapter 2. In the case where portfolio sets are constrained the absence of arbitrage opportunities at the individual level will differ from that at the aggregate level. As outlined in Angeloni and Cornet [1] we have the following definition.

**Definition 4.1.1** *Given the financial structure  $\mathcal{F} = (J, (\xi(j), V^j)_{j \in \mathcal{J}}, (Z^i)_{i \in \mathcal{I}})$ , the portfolio  $\bar{z}^i \in Z^i$  is said to have no arbitrage opportunities or to be arbitrage-free for agent  $i \in \mathcal{I}$  at the price  $q \in \mathbb{R}^{\mathbb{J}}$  if there is no portfolio  $z^i \in Z^i$  such that  $W_{\mathcal{F}}(q)z^i > W_{\mathcal{F}}(q)\bar{z}^i$ , that is,  $[W_{\mathcal{F}}(q)z^i](\xi) \geq [W_{\mathcal{F}}(q)\bar{z}^i](\xi)$ , for every  $\xi \in \mathcal{D}$ , with at least one strict inequality, or, equivalently, if*

$$W_{\mathcal{F}}(q) (Z^i - \bar{z}^i) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}.$$

*We say  $q$  is an arbitrage free asset price or the financial structure  $\mathcal{F}$  is said to be arbitrage-free at  $(q)$  if there exists no portfolios  $z^i \in Z^i$  ( $i \in \mathcal{I}$ ) such that  $W_{\mathcal{F}}(q)(\sum_{i \in \mathcal{I}} z^i) > 0$ , or, equivalently, if:*

$$W_{\mathcal{F}}(q) \left( \sum_{i \in \mathcal{I}} Z^i \right) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}.$$

Let the financial structure  $\mathcal{F}$  be arbitrage-free at  $q$ , and let  $\bar{z}^i \in Z^i$  ( $i \in \mathcal{I}$ ) such that  $\sum_{i \in \mathcal{I}} W_{\mathcal{F}}(q)\bar{z}^i = 0$ , then it is easy to see that, for every  $i \in \mathcal{I}$ ,  $\bar{z}^i$  is arbitrage-free at  $q$ . The converse is true, when  $\sum_{i \in \mathcal{I}} W_{\mathcal{F}}(q)Z^i \subset \text{cone} [\bigcup_{i \in \mathcal{I}} W_{\mathcal{F}}(q)(Z^i - \bar{z}^i)]$ . The later is true in particular when some agent's portfolio set is unconstrained, that is,  $Z^i = \mathbb{R}^{\mathbb{J}}$  for some  $i \in \mathcal{I}$ .

Consider the following non-satiation assumption:

**Assumption NS** ( $i$ ) For every  $\bar{x} \in \prod_{i \in \mathcal{I}} X^i$  such that  $\sum_{i \in \mathcal{I}} \bar{x}^i = \sum_{i \in \mathcal{I}} e^i$ ,

(Non-Satiation at Every Node) for every  $\xi^i \in \mathcal{D}$ , there exists  $x \in \prod_{i \in \mathcal{I}} X^i$  such that, for each  $\xi \neq \xi^i$ ,  $x^i(\xi) = \bar{x}^i(\xi)$  and  $x^i \in P^i(\bar{x})$ ;

(ii) if  $x^i \in P^i(\bar{x})$ , then  $[x^i, \bar{x}^i[ \subset P^i(\bar{x})$ .

It is well known that if preferences are non-satiated then there is no arbitrage at the individual level. In particular, under (NS), if  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  is an equilibrium of the economy  $(\mathcal{E}, \mathcal{F})$ , then  $\bar{z}^i$  is arbitrage-free at  $\bar{q}$  for every  $i \in \mathcal{I}$  (see Angeloni and Cornet [1]).

However, the set of asset prices at which there is no arbitrage at the individual level is larger than the set of asset prices that do not offer arbitrage opportunities at the aggregate level. We thus show that under some conditions on the space of income transfers, any no-arbitrage price at the aggregate level can be supported as an equilibrium price.

## 4.2 Existence of equilibrium

### 4.2.1 The main existence result

We will prove that when agents' portfolio sets are constrained, any asset price that is arbitrage free at the aggregate level can be characterized as an equilibrium asset price. Our approach however does not cover the general case of real assets which needs a different treatment. Let us consider, the financial economy (as described in Chapter 2),

$$(\mathcal{E}, \mathcal{F}) = [\mathcal{D}, \mathcal{H}, \mathcal{I}, (X^i, P^i, e^i)_{i \in \mathcal{I}}; \mathcal{J}, (\xi(j), V^j)_{j \in \mathcal{J}}, (Z^i)_{i \in \mathcal{I}}].$$

Define the set of attainable consumptions by

$$\hat{X} = \left\{ x \in \prod_{i \in \mathcal{I}} X^i \mid \sum_{i \in \mathcal{I}} x^i = \sum_{i \in \mathcal{I}} e^i \right\}$$

and for each  $i \in \mathcal{I}$ , let  $\widehat{X}^i$  be the projection of  $\widehat{X}$  on  $X^i$ .

We introduce the following assumptions.

**Assumption (C) (Consumption Side)** For all  $i \in \mathcal{I}$  and all  $\bar{x} \in \prod_{i \in \mathcal{I}} X^i$ ,

- (i)  $X^i$  is a closed and convex subset of  $\mathbb{R}^{\mathbb{L}}$  and  $\widehat{X}^i$  is compact<sup>1</sup> in  $\mathbb{R}^{\mathbb{L}}$ ;
- (ii) the preference correspondence  $P^i : \prod_{i \in \mathcal{I}} X^i \rightarrow X^i$ , is lower semicontinuous<sup>2</sup> and  $P^i(\bar{x})$  is convex;
- (iii) for every  $x^i \in P^i(\bar{x})$  for every  $(x')^i \in X^i$ ,  $(x')^i \neq x^i$ ,  $[(x')^i, x^i] \cap P^i(\bar{x}) \neq \emptyset$ ;<sup>3</sup>
- (iv) (Irreflexivity)  $\bar{x}^i \notin P^i(\bar{x})$ ;
- (v) (Non-Satiation of Preferences at Every Node) if  $\sum_{i \in \mathcal{I}} \bar{x}^i = \sum_{i \in \mathcal{I}} e^i$ , for every  $\xi \in \mathcal{D}$  there exists  $x \in \prod_{i \in \mathcal{I}} X^i$  such that, for each  $\xi' \neq \xi$ ,  $x^i(\xi') = \bar{x}^i(\xi')$  and  $x^i \in P^i(\bar{x})$ ;
- (vi) (Strong Survival Assumption)  $e^i \in \text{int} X^i$ .

**Assumption (F) (Financial Side)** Given an asset price  $q \in \mathbb{R}^J$ ,

- (i) for every  $i \in \mathcal{I}$ ,  $Z^i$  is closed, convex and contains zero.
- (ii) for every  $i \in \mathcal{I}$ ,  $W(q)Z^i$  is closed and convex;

---

<sup>1</sup>Note:  $\widehat{X}^i$  is compact if  $X^i$  is bounded below.

<sup>2</sup>A correspondence  $\varphi : X \longrightarrow Y$  is said to be lower semicontinuous at  $x_0 \in X$  if, for every open set  $V \subset Y$  such that  $V \cap \varphi(x_0)$  is not empty, there exists a neighborhood  $U$  of  $x_0$  in  $X$  such that, for all  $x \in U$ ,  $V \cap \varphi(x)$  is nonempty. The correspondence  $\varphi$  is said to be lower semicontinuous if it is lower semicontinuous at each point of  $X$ .

<sup>3</sup>This is satisfied, in particular, when  $P^i(\bar{x})$  is open in  $X^i$  (for its relative topology).

We can now state the main theorem characterizing equilibrium asset prices with arbitrage free asset prices under the appropriate condition on the transfer space.

**Theorem 4.2.1** *Suppose the financial exchange economy  $(\mathcal{E}, \mathcal{F})$  satisfies **C** and **F**. Let  $\bar{q} \in \mathbb{R}^J$  satisfy the following conditions <sup>4</sup>:*

- (i) *[(AF 2): Arbitrage-free at  $\bar{q}$ ]  $W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$*
- (ii) *[(W 6): ]  $\text{Span}(W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i)) \subset \text{cone} [\bigcup_{i \in \mathcal{I}} W(\bar{q})(Z^i)]$*
- (iii) *(WEQ)  $[-\sum_{i \in \mathcal{I}} Z^i] \cap \text{Ker } W(\bar{q}) \subset \sum_{i \in \mathcal{I}} [A(Z^i) \cap \text{Ker } W(\bar{q})]$*

*Then there exists  $(\bar{x}, \bar{z}, \bar{p})$  with  $\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}$  such that  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  is an equilibrium.*

## 4.2.2 Existence of a weak equilibrium

The above result will be proved as a consequence of the following more general result, which is interesting by itself.

**Definition 4.2.1** *A weak equilibrium in the economy  $(\mathcal{E}, \mathcal{F})$  is a list  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  satisfying condition (a) and (b) in Definition 2.2.1 and the following:*

$$(c') \sum_{i \in \mathcal{I}} W(\bar{q}) \bar{z}^i = 0.$$

Note that if we remove redundant assets as is done in many models of this kind (e.g. Cass [13]), then every weak equilibrium will be an equilibrium.

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<sup>4</sup>Given a convex set  $Y \subset \mathbb{R}^n$ , the asymptotic cone of  $Y$  is  $A(Y) := \{t \in \mathbb{R}^n \mid y + t \in Y, \forall y \in Y\}$ .

**Theorem 4.2.2** *Suppose the financial exchange economy  $(\mathcal{E}, \mathcal{F})$  satisfies **C** and **F**. Let  $\bar{q} \in \mathbb{R}^J$  satisfy the following conditions:*

- (i) *[(AF 2): Arbitrage-free at  $\bar{q}$ ]  $W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$*
- (ii) *[(W 6)]  $\text{Span}(W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i)) \subset \text{cone}[\bigcup_{i \in \mathcal{I}} W(\bar{q})(Z^i)]$ .*

*Then there exists  $(\bar{x}, \bar{z}, \bar{p})$  with  $\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}$  such that  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  is a weak equilibrium.*

The proof of Theorem 4.2.1 is then a consequence of the following proposition due to Da-Rocha and Triki [54].

**Proposition 4.2.1** *Existence of a weak equilibrium  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  implies the existence of an equilibrium if we have:*

$$[-\sum_{i \in \mathcal{I}} Z^i] \cap \text{Ker } W(\bar{q}) \subset \sum_{i \in \mathcal{I}} [A(Z^i) \cap \text{Ker } W(\bar{q})].$$

The condition in Proposition 4.2.1 holds if the following is true:

$$\text{Ker } W \subset \bigcup_{i \in \mathcal{I}} A(Z^i).$$

## 4.3 Arbitrage and income transfers

### 4.3.1 Concept of no-arbitrage at the aggregate level

Various notions of arbitrage free asset prices, found in the literature are listed below. Given an asset price  $\bar{q} \in \mathbb{R}^J$ ,

$$(AF 1) \text{ Im } (W(\bar{q})) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$$

$$(AF 2) W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$$

$$(AF\ 3) \quad W(\bar{q})A(\sum_{i \in \mathcal{I}}(Z^i)) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$$

$$(AF\ 4) \quad W(\bar{q})(\bigcup_{i \in \mathcal{I}} A(Z^i)) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$$

If there were some agent with an unconstrained portfolio set then all the above notions coincide and  $(AF\ 1)$  would suffice to describe absence of arbitrage at the aggregate level. Magill and Quinzii [53] use this notion. Da-Rocha and Triki [54] say that the payoff operator is arbitrage free under  $(AF\ 1)$ .

Under constrained portfolio sets  $AF\ 2$  would be more a more appropriate notion of absence of arbitrage in the aggregate level. This notion, due to Angeloni and Cornet [1], is considered in this paper.

The conditions  $AF\ 3$  and  $AF\ 4$  are weaker notions of absence of arbitrage.  $AF\ 3$  says that there is no infinite arbitrage in the aggregate level.  $AF\ 4$  says that no agent by herself can find an infinite arbitrage at  $\bar{q}$ . Da-Rocha and Triki [54] say that financial markets are arbitrage free under this condition. The relationship between these conditions is given by the following proposition, the proof of which is immediate.

**Proposition 4.3.1** *Given  $\bar{q} \in \mathbb{R}^{\mathbb{J}}$ , we have the following:*

$$AF\ 1 \implies AF\ 2 \implies AF\ 3 \implies AF\ 4$$

### 4.3.2 The transfer space condition

Cass [13] makes the observation that if all agents have constrained portfolio sets, then in order to ensure the existence of an equilibrium there must be some agent Mr.  $i_0$  such that:

$$\bigcup_{i \in \mathcal{I}} Z^i \subset Z^{i_0} \tag{4.1}$$



This condition states that there must be some agent who can accommodate the entire excess demand on asset markets. It is important to notice here is that with only nominal assets the agents problem of choosing a consumption stream and a portfolio can actually be viewed as two different choice problems. This due to the fact the income transfers on the financial markets do not depend on the commodity spot prices. On the consumption side each agent plans to choose an optimal consumption given commodity prices and the possible income transfers through financial markets. On the financial side each agent wishes to make the optimal income transfers given the asset prices. Thus all that matters to each agent, given the asset prices, is the set of possible income transfers that will enable to help her get the ‘best’ possible consumption stream. This point is well explained in Balasko, Cass and Siconolfi [6]. Since this is effectively a condition on the set of possible income transfers, hence its name ‘*Transfer space condition*’.

This condition has been assumed and used in the proof of existence when agents face linear constraints on their portfolio sets (the  $Z^i$  are strict subspaces of  $\mathbb{R}^J$ ). For instance Balasko, Cass and Siconolfi [6] and Cass, Siconolfi and Villanacci [16]. In fact in Balasko, Cass and Siconolfi [6], they also assume the dimension of this space of transfers for each agent is equal to the dimension of the corresponding portfolio set (a subspace) generating it.

In fact Cass [13] conjectures that this condition will ensure existence even when agents have general portfolio sets that are closed, convex and contain zero. However, in the case of such general portfolio sets, it turns out that a slightly different sort of condition is needed. Using Cass’ [13] conjecture as a motivation, we can modify this requirement as follows. Instead of the union

of portfolio sets on the left hand side, we actually need the span of the income transfers from the aggregate portfolio set. These two coincide in case where portfolio sets are linear. On the right hand side we can actually weaken the condition to the cone generated by the income transfers from the portfolio set of some agent  $\text{Mr. } i_0$ . This condition is stated as (W 6) in the statement of Theorem 4.2.2 and Theorem 4.2.1. Da-Rocha and Triki [54], use a stronger transfer space condition which is stated as (W 5) below.

There are several versions of this transfer space condition in terms of the portfolios sets or the possible income transfers that can be assumed, in order to prove existence of an equilibrium. These are listed below. Given  $\bar{q} \in \mathbb{R}^J$ :

$$(W\ 1) \ \exists i \in \mathcal{I} \text{ such that } \mathbb{R}^{\mathbb{D}} = W(\bar{q})Z^i$$

$$(W\ 1s) \ \exists i \in \mathcal{I} \text{ such that } Z^i = \mathbb{R}^{\mathbb{J}}$$

$$(W\ 2) \ \text{Im} (W(\bar{q})) \subset \bigcup_{i \in \mathcal{I}} W(\bar{q})(Z^i)$$

$$(W\ 2s) \ \mathbb{R}^{\mathbb{J}} \subset \bigcup_{i \in \mathcal{I}} Z^i$$

$$(W\ 3) \ \text{Span} (W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i)) \subset \bigcup_{i \in \mathcal{I}} W(\bar{q})(Z^i)$$

$$(W\ 3s) \ \text{Span} (\sum_{i \in \mathcal{I}} Z^i) \subset \bigcup_{i \in \mathcal{I}} Z^i$$

$$(W\ 4) \ \mathbb{R}^{\mathbb{D}} \subset \text{cone} [\bigcup_{i \in \mathcal{I}} W(\bar{q})(Z^i)]$$

$$(W\ 4s) \ \exists i \in \mathcal{I} \text{ such that } 0 \in \text{int} (Z^i)$$

$$(W\ 5) \ \text{Im} (W(\bar{q})) \subset \text{cone} [\bigcup_{i \in \mathcal{I}} W(\bar{q})(Z^i)]$$

$$(W\ 5s) \ \mathbb{R}^{\mathbb{J}} \subset \text{cone} [\bigcup_{i \in \mathcal{I}} Z^i]$$

$$(W\ 6) \ \text{Span} (W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i)) \subset \text{cone} [\bigcup_{i \in \mathcal{I}} W(\bar{q})(Z^i)]$$

$$(W\ 6s) \text{ Span } (\sum_{i \in \mathcal{I}} Z^i) \subset \text{cone } [\bigcup_{i \in \mathcal{I}} Z^i]$$

$$(W\ 7) \text{ Span } (W(\bar{q})A(\sum_{i \in \mathcal{I}} Z^i)) \subset \text{cone } [\bigcup_{i \in \mathcal{I}} W(\bar{q})(Z^i)]$$

$$(W\ 7s) \text{ Span } [A(\sum_{i \in \mathcal{I}} Z^i)] \subset \text{cone } [\bigcup_{i \in \mathcal{I}} Z^i]$$

In the above, each condition with a subscript  $s$  is a stronger version of the condition without the subscript. For instance  $W1s$  is stronger  $W1$ , and so on. Moreover the following proposition explains the relationship between these conditions, the proof of which is immediate.

**Proposition 4.3.2** *Given  $\bar{q} \in \mathbb{R}^{\mathbb{J}}$  we have the following:*

$$(a) \ W\ 1s \implies W\ 2s \implies W\ 3s \implies W\ 6s$$

$$(b) \ W\ 1s \implies W\ 4s \implies W\ 5s \implies W\ 6s$$

$$(c) \ W\ 1 \implies W\ 2 \implies W\ 3 \implies W\ 6$$

$$(d) \ W\ 1 \implies W\ 4 \implies W\ 5 \implies W\ 6$$

In order to show that any asset price at which  $(AF\ 4)$  (the most general notion of no-arbitrage with restricted participation) holds, can be supported as an equilibrium asset price, a stronger transfer space condition is required. The following proposition explains the reason for this.

**Proposition 4.3.3** *Given  $\bar{q} \in \mathbb{R}^{\mathbb{J}}$ , we have the following:*

$$(i) \ (Da-Rocha \text{ and Triki } [54]) \ AF\ 4 \text{ and } W\ 2 \implies AF\ 1 \text{ and } W\ 5$$

$$(ii) \ AF\ 4 \text{ and } W\ 3 \implies AF\ 2 \text{ and } W\ 6$$

**Proof of Proposition 4.3.3 (ii):**

By Proposition 4.3.2 we have  $W\ 3 \Rightarrow W\ 6$ . By contradiction suppose  $AF\ 3$  and  $W\ 3$  hold but  $W\ 6$  does not hold. Then there exists  $w \in \text{Span } W \sum_{i \in \mathcal{I}} Z^i$  such that  $w > 0$ . For all  $t \in \mathbb{N}$ ,  $tw \in \text{Span } W \sum_{i \in \mathcal{I}} Z^i \subset \cup_{i \in \mathcal{I}} WZ^i$ . Since the set of agents is finite, there exists  $i \in \mathcal{I}$  and an increasing sequence of integers  $(t_n)_n$  such that  $t_n w \in WZ^i$  for each  $n \in \mathbb{N}$ . Let  $w^i \in WZ^i$  then

$$(1 - \frac{1}{t_n})w^i + \frac{1}{t_n}t_n w \in WZ^i$$

Passing to the limit we have  $w^i + w \in WZ^i$ . Thus  $w \in A(WZ^i)$ . Contradiction with  $AF\ 3$ .  $\square$

#### 4.4 Some consequences of the main theorem

Many of the previous results in the literature can now be states as corollaries of the central theorems in this chapter, Theorem 4.2.2 and Theorem 4.2.1. In view of the theorems in this chapter and Proposition 4.3.1, Proposition 4.3.2 and Proposition 4.3.3 we have the following consequences.

**Corollary 4.4.1 Unrestricted Case 1: (Cass [13])** *If the conditions in Theorem 4.2.2 are replaced with:*

$$(AF\ 2) \text{ [Arbitrage-free at } \bar{q}] \quad W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$$

$$(W\ 1s) \quad \exists i \in \mathcal{I} \text{ such that } Z^i = \mathbb{R}^{\mathbb{J}}$$

*Then there exists  $(\bar{x}, \bar{z}, \bar{p})$  with  $\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}$  such that  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  is a weak equilibrium.*

**Corollary 4.4.2 (Angeloni-Cornet [1])** *If the conditions in Theorem 4.2.2 are replaced with:*

$$(AF\ 2) \text{ [Arbitrage-free at } \bar{q}] \quad W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$$

$$(W\ 4s) \quad \exists i \in \mathcal{I} \text{ such that } 0 \in \text{int} (Z^i)$$

*Then there exists  $(\bar{x}, \bar{z}, \bar{p})$  with  $\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}$  such that  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  is a weak equilibrium.*

**Corollary 4.4.3** *If the conditions in Theorem 4.2.2 are replaced with:*

$$(AF\ 2) \text{ [Arbitrage-free at } \bar{q}] \quad W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$$

$$(W\ 6s) \quad \text{Span} (\sum_{i \in \mathcal{I}} Z^i) \subset \text{cone} [\bigcup_{i \in \mathcal{I}} (Z^i)]$$

*Then there exists  $(\bar{x}, \bar{z}, \bar{p})$  with  $\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}$  such that  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  is a weak equilibrium.*

**Corollary 4.4.4 (Martins Da-Rocha and Triki 1 [54])** *If the conditions in Theorem 4.2.2 are replaced with:*

$$(AF\ 1) \text{ [Arbitrage-free at } \bar{q}] \quad \text{Im} (W(\bar{q})) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$$

$$(W\ 5) \quad \text{Im} (W(\bar{q})) \subset \text{cone} [\bigcup_{i \in \mathcal{I}} W(\bar{q})(Z^i)]$$

*Then there exists  $(\bar{x}, \bar{z}, \bar{p})$  with  $\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}$  such that  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  is a weak equilibrium.*

Following are some more consequences when the more general notion of no-arbitrage (AF 4) is used. Notice that the transfer space condition is stronger.

**Corollary 4.4.5 (Unrestricted Case 2)** *If the conditions in Theorem 4.2.2 are replaced with:*

$$(AF\ 4) \text{ [Arbitrage-free at } \bar{q}] \quad W(\bar{q})(\bigcup_{i \in \mathcal{I}} A(Z^i)) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$$

$$(W\ 1s) \quad \exists i \in \mathcal{I} \text{ such that } Z^i = \mathbb{R}^{\mathbb{J}}$$

*Then there exists  $(\bar{x}, \bar{z}, \bar{p})$  with  $\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}$  such that  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  is a weak equilibrium.*

**Corollary 4.4.6 (Martins Da-Rocha and Triki 2 [54])** *If the conditions in Theorem 4.2.2 are replaced with:*

$$(AF\ 4) \text{ [Arbitrage-free at } \bar{q}] \quad W(\bar{q})(\bigcup_{i \in \mathcal{I}} A(Z^i)) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$$

$$(W\ 2) \quad Im(W(\bar{q})) \subset \bigcup_{i \in \mathcal{I}} W(\bar{q})(Z^i)$$

*Then there exists  $(\bar{x}, \bar{z}, \bar{p})$  with  $\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}$  such that  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  is a weak equilibrium.*

**Corollary 4.4.7** *If the conditions in Theorem 4.2.2 are replaced with:*

$$(AF\ 4) \text{ [Arbitrage-free at } \bar{q}] \quad W(\bar{q})(\bigcup_{i \in \mathcal{I}} A(Z^i)) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$$

$$(W\ 3) \quad Span(W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i)) \subset \bigcup_{i \in \mathcal{I}} W(\bar{q})(Z_i)$$

*Then there exists  $(\bar{x}, \bar{z}, \bar{p})$  with  $\bar{p}(\xi) \neq 0, \forall \xi \in \mathcal{D}$  such that  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  is a weak equilibrium.*

#### 4.4.1 Counterexample 1:

It is possible to find a meaningful example where (AF 2) and (W 6) does not imply (AF 1) and (W 5). Thus the set of asset prices used to characterize weak equilibrium asset prices in Theorem 4.2.2 is larger than that in Da-Rocha and Triki [54].

Let  $\mathcal{T} = \{0, 1\}$ ,  $\mathcal{D} = \{0, 1, 2\}$  and  $\mathcal{I} = \{1, 2\}$ . Let  $\mathcal{J} = \{1, 2\}$  such that for all  $j \in \mathcal{J}$ ,  $\xi(j) = 0$ . The payoffs are given by  $V^1 = (2, 1)$  and  $V^2 = (2, 2)$ . Consider the asset price  $\bar{q} = (2, 2)$ . Then the total payoff matrix is given by:

$$W(\bar{q}, V) = \begin{bmatrix} -2 & -2 \\ 2 & 2 \\ 1 & 2 \end{bmatrix}$$

Let

$$Z^1 = Z^2 = \{\alpha(1, 1) \mid \alpha \in [-1, 1]\}$$

then

$$W(\bar{q}, V)Z^1 = W(\bar{q}, V)Z^2 = \{\alpha(-4, 4, 3) \mid \alpha \in [-1, 1]\}$$

$$W(\bar{q}, V)(\sum_{i \in \mathcal{I}} Z^i) = \{\alpha(-8, 8, 6) \mid \alpha \in [-1, 1]\}$$

$$\text{span} [W(\bar{q}, V)(\sum_{i \in \mathcal{I}} Z^i)] = \text{cone} (\cup_{i \in \mathcal{I}} W(\bar{q}, V)Z^i)$$

At this price  $\bar{q}$ , (AF 2) and (W 6) are satisfied but not (AF 1) and (W 5). This price  $\bar{q}$  can be characterized as an equilibrium asset price according to Theorem 4.2.2.

For instance suppose there is only one good available in each state and the two agents have the following endowments and utility functions:

$$e^1 = (2, 6, 6) \text{ and } u^1(x^1(0), x^1(1), x^1(2)) = 2x^1(0) + x^1(1) + x^1(2)$$

$$e^2 = (4, 4, 3) \text{ and } u^2(x^2(0), x^2(1), x^2(2)) = x^2(0) + 2x^2(1) + 2x^2(2)$$

Each agent  $i$ 's income transfer potential is determined by  $\alpha^i \in [-1, 1]$ . After incorporating the budget constraints, we can then write the reduced form utility functions in terms of the choice of portfolio  $\alpha^i$ .

$$\hat{u}^1(\alpha^1) = 16 - \alpha^1$$

$$\hat{u}^2(\alpha^2) = 18 + 10\alpha^2$$

The agents can then be viewed as maximizing their reduced form utilities over their choice of  $\alpha^i \in [-1, 1]$ . So  $\alpha^1 = -1$  maximizes agent 1's utility and  $\alpha^2 = 1$  maximizes agent 2's utility.

Thus  $\bar{x}^1 = (6, 2, 3)$ ;  $\bar{z}^1 = (-1, -1)$  and  $\bar{x}^2 = (0, 8, 6)$ ;  $\bar{z}^2 = (1, 1)$  along with  $\bar{q} = (2, 2)$  is an equilibrium.  $\square$

Modifying the above example so that the transfer space requirement (W 6) does not hold, the impossibility of the characterization result is given in the following example.

#### 4.4.2 Counterexample 2:

Here we show that the conditions in Theorem 4.2.2 are fairly tight. That is if AF 2 and W 6 do not hold at some price  $\bar{q}$ , then  $\bar{q}$  cannot be an equilibrium asset price.



Consider the example above with the following modification of the agents' portfolio sets:

$$Z^1 = Z^2 = \{\alpha(1, 1) \mid \alpha \in [0, 1]\}$$

then

$$W(\bar{q}, V)Z^1 = W(\bar{q}, V)Z^2 = \{\alpha(-4, 4, 3) \mid \alpha \in [0, 1]\}$$

$$W(\bar{q}, V)(\sum_{i \in \mathcal{I}} Z^i) = \{\alpha(-8, 8, 6) \mid \alpha \in [0, 1]\}$$

The same price in the previous example  $\bar{q}$  does not provide arbitrage opportunities at the aggregate level, since

$$W(\bar{q}, V)(\sum_{i \in \mathcal{I}} Z^i) \cap \mathbb{R}_+^{\mathbb{D}} = \{0\}$$

However the transfer space condition does not hold, since

$$\text{span } W(\bar{q}, V)(\sum_{i \in \mathcal{I}} Z^i) = \{\alpha(-4, 4, 3) \mid \alpha \in \mathbb{R}\}$$

is not contained in

$$\text{cone}[\bigcup_{i \in \mathcal{I}} W(\bar{q}, V)Z^i] = \{\alpha(-4, 4, 3) \mid \alpha \in \mathbb{R}_+\}$$

Again working with the reduced form utility functions as in the previous example, note that for accounts clearing in the asset market at  $\bar{q}$  we need  $\alpha^1 = -\alpha^2$ . Given  $Z^1$  and  $Z^2$ , this is possible only when  $\alpha^1 = \alpha^2 = 0$ . However, from agent 2's problem in the previous example we see that  $\alpha^2 = 1$  is feasible within her budget restrictions and maximizes her reduced form utility. Thus  $q = (2, 2)$  cannot be an equilibrium asset price.  $\square$

## 4.5 Proof of main theorem

### 4.5.1 Proof of Theorem 4.2.2 (under additional assumptions):

#### Additional assumptions (K)

- (i) The sets  $X^i$  and  $W(\bar{q})Z^i$  are bounded;
- (ii)[Local Non-Satiation] for every  $\bar{x} \in \prod_{i \in \mathcal{I}} X^i$ , for every  $x^i \in P^i(\bar{x})$  then  $[x^i, \bar{x}^i[ \subset P^i(\bar{x})$ .

Before entering the proof of Theorem 4.2.2 we will state and prove the following:

**Lemma 4.5.1** *If Conditions (i) and (ii) in Theorem 4.2.2 hold at  $\bar{q}$  then under Assumption (K) there exists  $\lambda \in \mathbb{R}_{++}^{\mathbb{D}}$  such that:*

$$W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i) \subset \lambda^\perp := \{t \in \mathbb{R}^{\mathbb{D}} \mid \lambda \bullet_{\mathbb{D}} t = 0\}$$

**Proof of Lemma 4.5.1:** For all  $i \in \mathcal{I}$ ,  $W(\bar{q})(Z^i)$  is compact (by **F** and **K(i)**).

Thus  $W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i)$  is compact and hence closed. Since  $\mathcal{F}$  is arbitrage free (Condition (i) in theorem) at  $\bar{q}$ , there exists  $\lambda \in \mathbb{R}_{++}^{\mathbb{D}}$  such that

$$\forall w \in \sum_{i \in \mathcal{I}} W(\bar{q})Z^i, \quad \lambda \bullet_{\mathbb{D}} w \leq 0.$$

Since  $\forall i \in \mathcal{I}, 0 \in W(\bar{q})Z^i$ , we have  $\forall w^i \in W(\bar{q})Z^i, \lambda \bullet_{\mathbb{D}} w^i \leq 0$ . By (ii)  $\forall w \in W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i), -w \in \text{cone} [\bigcup_{i \in \mathcal{I}} W(\bar{q})(Z^i)]$ . Thus  $\exists k \in \mathcal{I}, \exists \alpha > 0$  such that  $-w = \alpha w^k$  for some  $w^k \in W(\bar{q})Z^k$ . Thus  $-\lambda \bullet_{\mathbb{D}} w = \lambda \bullet_{\mathbb{D}} (\alpha w^k) \leq 0$ .  $\square$

To simplify the notation, in the following we will suppress the dependence of  $W$  on  $\bar{q}$ .

## Preliminaries

Define the following<sup>5</sup>:

$$B = \{p \in \mathbb{R}^{\mathbb{L}} \mid \|\lambda \square p\| \leq 1\}$$

$$\rho(p) = 1 - \|\lambda \square p\|$$

Let  $\mathbf{1} = (1, \dots, 1)$  denote the element in  $\mathbb{R}^{\mathbb{D}}$ , whose coordinates are all equal to one.

Let  $\Gamma$  denote the space of continuous functions from  $B$  to  $\mathbb{R}^{\mathbb{D}}$ . For every  $\gamma \in \Gamma$  we have  $\gamma = (\gamma(p, \xi))_{\xi \in \mathcal{D}}$ .

Given  $p \in B$  and  $\gamma \in \Gamma$ , for all  $i \in \mathcal{I}$  define:

$$\begin{aligned} \beta_{\gamma}^i(p) &= \left\{ (x^i, w^i) \in X^i \times WZ^i : \exists \tau^i \in [0, 1], p \square (x^i - e^i) \leq w^i + \tau^i \gamma(p) + \rho(p) \mathbf{1} \right\}, \\ \alpha_{\gamma}^i(p) &= \left\{ (x^i, w^i) \in X^i \times WZ^i : \exists \tau^i \in [0, 1], p \square (x^i - e^i) \ll w^i + \tau^i \gamma(p) + \rho(p) \mathbf{1} \right\}. \end{aligned}$$

Using the procedure outlined in Da-Rocha and Triki [54] we choose  $\gamma \in \Gamma$  as in the following lemma, and drop the subscript  $\gamma$  from the above sets  $\beta^i(p)$  and  $\alpha^i(p)$ .

**Lemma 4.5.2** *There exists a continuous mapping  $\gamma : B \rightarrow \mathbb{R}^{\mathbb{D}}$  such that:*

$$(i) \quad \forall p \in B, \lambda \bullet_{\mathbb{D}} \gamma(p) = 0$$

$$(ii) \quad \forall p \in B, \forall w \in W(\sum_{i \in \mathcal{I}} Z^i), w \bullet_{\mathbb{D}} \gamma(p) = 0$$

$$(iii) \quad \forall p \in B, \bigcup_{i \in \mathcal{I}} \alpha_{\gamma}^i(p) \neq \emptyset$$

---

<sup>5</sup>For  $x \in \mathbb{R}^n$ ,  $\|x\|$  denotes the euclidean norm.

**Proof of Lemma 4.5.2:** Define the following subsets of  $\Gamma$ :

$$\Gamma_1 := \{\gamma \in \Gamma \mid \forall p \in B, \gamma(p) \in \lambda^\perp\}$$

$$\Gamma_2 := \{\gamma \in \Gamma \mid \forall p \in B, \gamma(p) \in [W(\sum_{i \in \mathcal{I}} Z^i)]^\perp\}$$

$$\Gamma_3 := \{\gamma \in \Gamma \mid \forall p \in B, \bigcup_{i \in \mathcal{I}} \alpha_\gamma^i(p) \neq \emptyset\}$$

We will show that  $\Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \neq \emptyset$ .

**Step(1):**  $\Gamma_1 \cap \Gamma_2 \neq \emptyset$

Define the set

$$\Delta = \{\delta \in \mathbb{R}^\mathbb{D} \mid \delta \in \lambda^\perp \cap [(W(\sum_{i \in \mathcal{I}} Z^i))]^\perp, \|\delta\| \leq 1\}$$

Notice from assumption **C** that  $\exists r > 0$  such that  $\forall i \in \mathcal{I}, U := B_{\mathbb{L}}(0, r) \subset X^i$ .

Define the correspondence  $\Psi$  from  $B$  to  $\Delta$  by

$$\Psi(p) = \{\delta \in \Delta \mid \exists u \in U, \exists w \in \text{Span}(W(\sum_{i \in \mathcal{I}} Z^i)), p \sqcup u << w + \delta + \rho(p)\mathbf{1}\}$$

We will show that there is a continuous selection  $\gamma$  of  $\Psi$  such that  $\gamma \in \Gamma_1 \cap \Gamma_2$ .

In order to do this we will use the continuous selection result in Proposition 1.5.3, Florenzano [28].

Firstly, notice that  $\forall p \in B, \Psi(p)$  is clearly convex valued. To see this, let  $\delta_1 \in \Psi(p)$  and  $\delta_2 \in \Psi(p)$ , then

$$\exists u_1 \in U, \exists w_1 \in \text{Span}(W(\sum_{i \in \mathcal{I}} Z^i)) : p \sqcup u_1 << w_1 + \delta_1 + \rho(p)\mathbf{1}$$

$$\exists u_2 \in U, \exists w_2 \in \text{Span}(W(\sum_{i \in \mathcal{I}} Z^i)) : p \sqcup u_2 << w_2 + \delta_2 + \rho(p)\mathbf{1}$$

Then for all  $\alpha \in [0, 1]$ ,  $(\alpha u_1 + (1-\alpha)u_2) \in U$ ,  $(\alpha w_1 + (1-\alpha)w_2) \in \text{Span } (W(\sum_{i \in \mathcal{I}} Z^i))$  and  $(\alpha \delta_1 + (1-\alpha)\delta_2) \in \Delta$ , and

$$p \square (\alpha u_1 + (1-\alpha)u_2) << (\alpha w_1 + (1-\alpha)w_2) + (\alpha \delta_1 + (1-\alpha)\delta_2) + \rho(p)\mathbf{1}$$

Thus for all  $p \in B$ ,  $\Psi(p)$  is convex.

Also, notice that  $\forall p \in B, \Psi(p) \neq \emptyset$ . To see this, let  $p \in B$ ,

Case(i)  $\rho(p) > 0$ : Let  $u = 0$  and  $w = 0$  then  $\delta = 0 \in \Psi(p)$ .

Case (ii)  $\rho(p) = 0$ : i.e.  $\sum_{\xi \in \mathcal{D}} \lambda(\xi)^2 \|p(\xi)\|^2 = 1$ . Since  $\lambda >> 0, \exists \xi \in \mathcal{D}$  such that  $p(\xi) \neq 0$ . Thus  $\exists u \in U$  such that  $p \square u < 0$ . Since  $\lambda >> 0, \exists t \in \lambda^\perp$  such that  $p \square u << t$ .

Since  $\mathbb{R}^\mathbb{D} = \text{Span } (W(\sum_{i \in \mathcal{I}} Z^i)) + [W(\sum_{i \in \mathcal{I}} Z^i)]^\perp$ , there exists  $w \in \text{Span } (W(\sum_{i \in \mathcal{I}} Z^i))$  and  $\delta \in [W(\sum_{i \in \mathcal{I}} Z^i)]^\perp$  such that  $t = w + \delta$ . Using Lemma 4.5.1 we can see that,

$$0 = \lambda \bullet_\mathbb{D} t = \lambda \bullet_\mathbb{D} w + \lambda \bullet_\mathbb{D} \delta = 0 + \lambda \bullet_\mathbb{D} \delta = \lambda \bullet_\mathbb{D} \delta$$

Thus  $\delta \in \lambda^\perp$ . Now we have

$$p \square u << w + \delta$$

For  $\tau > 0$  small enough

$$p \square (\tau u) << \tau w + \tau \delta$$

with  $\tau u \in U, \tau w \in \text{Span } (W(\sum_{i \in \mathcal{I}} Z^i))$  and  $\|\tau \delta\| \leq 1$ . Hence  $\tau \delta \in \Psi(p)$ .

Thus  $\Psi(p) \neq \emptyset$ . In view of Case (i) and Case (ii),  $\forall p \in B, \Psi(p) \neq \emptyset$ .

Now we can show that  $\Psi$  is lower semicontinuous on  $B$ .

Denote the graph of  $\Psi$  by,

$$G_\Psi : \{(p, \delta) \in B \times \Delta \mid \delta \in \Psi(p)\}$$

We will show  $G_\Psi$  is open and hence  $\Psi$  is l.s.c. In fact we will show  $(B \times \Delta) \setminus G_\Psi$  is closed. Let  $\{(p_n, \delta_n)\} \in (B \times \Delta) \setminus G_\Psi$  and  $(p_n, \delta_n) \rightarrow (p, \delta)$ . By contradiction suppose  $(p, \delta) \in G_\Psi$ . Then  $\exists \bar{u} \in U, \exists \bar{w} \in \text{Span}(W(\sum_{i \in \mathcal{I}} Z^i))$  such that

$$\forall \xi \in \mathcal{D}, \quad p(\xi) \bullet_{\mathbb{H}} \bar{u}(\xi) << \bar{w}(\xi) + \delta(\xi) + \rho(p) \quad (*)$$

Also  $\forall n, \delta_n \notin \Psi(p_n)$  thus  $\forall n, \forall u \in U$  and  $\forall w \in \text{Span}(W(\sum_{i \in \mathcal{I}} Z^i))$

$$\exists \xi \in \mathcal{D} \text{ such that } p_n(\xi) \bullet_{\mathbb{H}} u(\xi) \geq w(\xi) + \delta_n(\xi) + \rho(p)$$

in particular,

$$\exists \xi \in \mathcal{D} \text{ such that } p_n(\xi) \bullet_{\mathbb{H}} \bar{u}(\xi) \geq \bar{w}(\xi) + \delta_n(\xi) + \rho(p)$$

Since  $p_n \rightarrow p, \rho(p_n) \rightarrow \rho(p)$  and  $\forall \xi \in \mathcal{D}, p_n(\xi) \rightarrow p(\xi)$ . Since  $\delta_n \rightarrow \delta, \forall \xi \in \mathcal{D}, \delta_n(\xi) \rightarrow \delta(\xi)$ . Thus in the limit

$$\exists \xi \in \mathcal{D} \text{ such that } p(\xi) \bullet_{\mathbb{H}} \bar{u}(\xi) \geq \bar{w}(\xi) + \delta(\xi) + \rho(p)$$

Contradiction with  $(*)$ . Thus  $\Psi$  is l.s.c. on  $B$ .

Applying Proposition 1.5.3 in Florenzano [28] we can conclude that there is a continuous selection  $\gamma$  of  $\Psi$  and  $\forall p \in B, \gamma(p) \in \lambda^\perp \cap [W(\sum_{i \in \mathcal{I}} Z^i)]^\perp$ . Thus  $\gamma \in \Gamma_1 \cap \Gamma_2$ .

**Step(2):**  $\Gamma_1 \cap \Gamma_2 \subset \Gamma_3$

Let  $\gamma$  be the continuous selection of  $\Psi$  obtained in Step (1) above. We need to show  $\forall p \in B, \exists k \in \mathcal{I}$  such that  $\alpha_\gamma^k(p) \neq \emptyset$ . To see this, let  $p \in B$

Case(i)  $\rho(p) > 0$ : for all  $i \in \mathcal{I}$  with  $\tau^i = 0$  we have  $(x^i, w^i) = (e^i, 0) \in \alpha_\gamma^i(p)$ .

Case(ii)  $\rho(p) = 0$ . Since  $\gamma(p) \in \Psi(p), \exists u \in U, \exists w \in \text{Span}(W(\sum_{i \in \mathcal{I}} Z^i))$  such that

$$p \square u << w + \gamma(p)$$

for  $\tau \in (0, 1]$  small enough

$$p \square (\tau u) << \tau w + \tau \gamma(p)$$

with  $(\tau u) \in U$ ,  $(\tau w) \in WZ^k$  for some  $k \in \mathcal{I}$  (by Condition (ii) in Theorem 4.2.2). Thus there exists  $k \in \mathcal{I}$  such that setting  $\tau^k = \tau$  we have  $(\tau u + e^k, \tau w) \in \alpha_\gamma^k(p)$ . Thus  $\gamma \in \Gamma_3$ .

In view of Step (1) and Step (2) we have  $\Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \neq \emptyset$ .  $\square$

## The fixed point argument

For  $(x, w, p) \in \prod_{i \in \mathcal{I}} X^i \times \prod_{i \in \mathcal{I}} WZ^i \times B$ , we define the correspondences  $\Phi^i$  for  $i \in \mathcal{I}_0 = \{0\} \cup \mathcal{I}$  as follows:

$$\Phi^0(x, w, p) = \left\{ (p') \in B \mid \lambda \square (p' - p) \bullet_{\mathbb{L}} \sum_{i \in \mathcal{I}} (x^i - e^i) > 0 \right\},$$

and for every  $i \in \mathcal{I}$ ,

$$\Phi^i(x, w, p) = \begin{cases} \{(e^i, 0)\} & \text{if } (x^i, w^i) \notin \beta^i(p) \text{ and } \alpha^i(p) = \emptyset, \\ \beta^i(p) & \text{if } (x^i, w^i) \notin \beta^i(p) \text{ and } \alpha^i(p) \neq \emptyset, \\ \alpha^i(p) \cap (P^i(x) \times WZ^i) & \text{if } (x^i, w^i) \in \beta^i(p). \end{cases}$$

The existence proof relies on the following fixed-point-type theorem due to Gale and MasCollé ([31]).

**Theorem 4.5.1** *Let  $\mathcal{I}_0$  be a finite set, let  $C^i$  ( $i \in \mathcal{I}_0$ ) be a nonempty, compact, convex subset of some Euclidean space, let  $C = \prod_{i \in \mathcal{I}_0} C^i$  and let  $\Phi^i$  ( $i \in \mathcal{I}_0$ ) be a correspondence from  $C$  to  $C^i$ , which is lower semicontinuous and convex-valued. Then, there exists  $\bar{c} \in C$  such that, for every  $i \in \mathcal{I}_0$  [either  $\bar{c}^i \in \Phi^i(\bar{c})$  or  $\Phi^i(\bar{c}) = \emptyset$ ].*

We now show that, the set  $C^0 = B$ , and for all  $i \in \mathcal{I}$ ,  $C^i = X^i \times Z^i$  and the above defined correspondences  $\Phi^i$  ( $i \in \mathcal{I}_0$ ) satisfy the assumptions of Theorem 4.5.1.

**Claim 4.5.1** *For every  $\bar{c} := (\bar{x}, \bar{w}, \bar{p}) \in \prod_{i \in \mathcal{I}} X^i \times \prod_{i \in \mathcal{I}} WZ^i \times B$ ,*

*(i)  $\Phi^i(\bar{c})$  is convex (possibly empty)*

*(ii)  $\bar{p} \notin \Phi^0(\bar{c})$ , and for all  $i \in \mathcal{I}$ ,  $(\bar{x}^i, \bar{w}^i) \notin \Phi^i(\bar{c})$*

*(iii) for every  $i \in \mathcal{I}_0$ , the correspondence  $\Phi^i$  is lower semicontinuous at  $\bar{c}$*

**Proof of Claim 4.5.1:** Let  $\bar{c} := (\bar{x}, \bar{y}, (\bar{p})) \in \prod_{i \in \mathcal{I}} X^i \times \prod_{i \in \mathcal{I}} WZ^i \times B$  be given.

**Proof of (i):** Clearly  $\Phi^0(\bar{c})$  is convex. For every  $i \in \mathcal{I}$ , recalling that  $P^i(\bar{x})$  and  $WZ^i$  are convex sets, by Assumption **(C)** and **(F)**, we have  $\Phi^i(\bar{c})$  is a convex set.

**Proof of (ii):** Clearly,  $(\bar{p}) \notin \Phi^0(\bar{c})$  and  $(\bar{x}^i, \bar{w}^i) \notin \Phi^i(\bar{c})$  follows from the definitions of these sets and the fact that  $\bar{x}^i \notin P^i(\bar{x})$  (from Assumption **(C)**).

**Proof of (iii):** We need to show  $\Phi^i$  is lower semicontinuous for all  $i \in \mathcal{I}_0$ . Since  $\Phi^0$  has an open graph, clearly it is lower semicontinuous. To show lower semicontinuity of  $\Phi^i$  for  $i \in \mathcal{I}$ , we will distinguish three cases:

*Case(1) :*  $(\bar{x}^i, \bar{w}^i) \notin \beta^i(\bar{p})$  and  $\alpha^i(\bar{p}) = \emptyset$ . Then  $\Phi^i(\bar{c}) = \{(e^i, 0)\} \subset U$ . The set  $\Omega^i = \{(x^i, w^i, p) \mid (x^i, w^i) \notin \beta^i(p)\}$  is an open subset of  $X^i \times WZ^i \times B$  (by Assumptions **(C)** and **(F)**). To see this, let  $\{(x_n, w_n, p_n)\}$  be such that  $(x_n, w_n) \in \beta^i(p_n)$  and  $(x_n, w_n, p_n) \rightarrow (x, w, p)$ . Since for all  $n$ ,  $(x_n, w_n) \in \beta^i(p_n)$ , there exists  $\tau_n \in [0, 1]$  such that

$$p_n \square (x_n - e^i) \leq w_n + \tau_n \gamma(p_n) + \rho(p_n) \mathbf{1}$$



In the limit we have

$$p \sqcap (x - e^i) \leq w + \tau \gamma(p) + \rho(p) \mathbf{1}$$

Where  $\tau = \lim_{n \rightarrow \infty} \tau_n \in [0, 1]$ . Thus  $(x, w) \in \beta^i(p)$ .

Thus  $\Omega^i$  contains an open neighborhood  $O$  of  $\bar{c}$ . Now, let  $c = (x, w, p) \in O$ . If  $\alpha^i(p) = \emptyset$  then  $\Phi^i(c) = \{(e^i, 0)\} \subset U$  and so  $\Phi^i(c) \cap U$  is nonempty. If  $\alpha^i(p) \neq \emptyset$  then  $\Phi^i(c) = \beta^i(p)$ . But Assumptions **(C)** and **(F)** imply that  $(e^i, 0) \in X^i \times WZ^i$ , hence  $(e^i, 0) \in \beta^i(p)$  ( with  $\tau^i = 0$  and noticing that  $\rho(p) \geq 0$ ). So  $\{(e^i, 0)\} \subset \Phi^i(c) \cap U$  which is also nonempty.

*Case(2)* :  $\bar{c} = (\bar{x}^i, \bar{w}^i, \bar{p}) \in \Omega^i := \{c = (x^i, w^i, p) : (x^i, w^i) \notin \beta^i(p) \text{ and } \alpha^i(p) \neq \emptyset\}$ . Then the set  $\Omega^i$  is clearly open (since its complement is closed).

On the set  $\Omega^i$  one has  $\Phi^i(c) = \beta^i(p)$ . We recall that  $\emptyset \neq \Phi^i(\bar{c}) \cap U = \beta^i(\bar{p}) \cap U$ . We notice that  $\beta^i(\bar{p}) = \text{cl } \alpha^i(\bar{p})$  since  $\alpha^i(\bar{p}) \neq \emptyset$ . Consequently,  $\alpha^i(\bar{p}) \cap U \neq \emptyset$  and we chose a point  $(x^i, w^i) \in \alpha^i(\bar{p}) \cap U$ , that is,  $(x^i, w^i) \in [X^i \times WZ^i] \cap U$  and for some  $\tau^i \in [0, 1]$ ,

$$\bar{p} \sqcap (x^i - e^i) \ll w^i + \tau^i \gamma(\bar{p}) + \rho(\bar{p}) \mathbf{1}.$$

Clearly the above inequality is also satisfied for the same point  $(x^i, w^i)$  and the same  $\tau^i$  when  $p$  belongs to a neighborhood  $O$  of  $\bar{p}$  small enough (using the continuity of  $\rho(\cdot)$  and  $\gamma(\cdot)$ ). This shows that on  $O$  one has  $\emptyset \neq \alpha^i(p) \cap U \subset \beta^i(p) \cap U = \Phi(c) \cap U$ .

*Case(3)* :  $(\bar{x}^i, \bar{w}^i) \in \beta^i(\bar{p})$ . By assumption we have

$$\emptyset \neq \Phi^i(\bar{c}) \cap U = \alpha^i(\bar{p}) \cap [P^i(\bar{x}) \times WZ^i] \cap U.$$

By an argument similar to what is done above, one shows that there exists an open neighborhood  $N$  of  $\bar{p}$  and an open set  $M$  such that, for every  $p \in N$ , one

has  $\emptyset \neq M \subset \alpha^i(p) \cap U$ . Since  $P^i$  is lower semicontinuous at  $\bar{c}$  (by Assumption (C)), there exists an open neighborhood  $\Omega$  of  $\bar{c}$  such that, for every  $c \in \Omega$ ,  $\emptyset \neq [P^i(x) \times WZ^i] \cap M$ , hence

$$\emptyset \neq [P^i(x) \times WZ^i] \cap \alpha^i(p) \cap U \subset \beta^i(p) \cap U, \text{ for every } c \in \Omega.$$

Consequently, from the definition of  $\Phi^i$ , we get  $\emptyset \neq \Phi^i(c) \cap U$ , for every  $c \in \Omega$ .

The correspondence  $\Psi^i := \alpha^i \cap (P^i \times WZ^i)$  is lower semicontinuous on the whole set, being the intersection of an open graph correspondence and a lower semicontinuous correspondence. Then there exists an open neighborhood  $O$  of  $\bar{c} := (\bar{x}, \bar{w}, \bar{p})$  such that, for every  $(x, w, p) \in O$ , then  $U \cap \Psi^i(x, w, p) \neq \emptyset$  hence  $\emptyset \neq U \cap \Phi^i(x, w, p)$  (since we always have  $\Psi^i(x, w, p) \subset \Phi^i(x, w, p)$ ).  $\square$

In view of Claim 4.5.1, we can apply the Gale-MasColell theorem. Then we have the following :

$$\forall p \in B, p \sqsupset \sum_{i \in \mathcal{I}} (\bar{x}^i - e^i) \leq \bar{p} \sqsupset \sum_{i \in \mathcal{I}} (\bar{x}^i - e^i) \quad (4.2)$$

$$\forall i \in \mathcal{I}, (\bar{x}^i, \bar{w}^i) \in \beta^i(\bar{p}) \text{ and } \alpha^i(\bar{p}) \cap (P^i(\bar{x}) \times WZ^i) = \emptyset. \quad (4.3)$$

$$\forall i \in \mathcal{I}, \exists \bar{z}^i \in Z^i \text{ such that } \bar{w}^i = W(\bar{q})\bar{z}^i. \quad (4.4)$$

**The list  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  is a weak equilibrium**

**Claim 4.5.2**  $\sum_{i \in \mathcal{I}} (\bar{x}^i - e^i) = 0$ .

**Proof of Claim 4.5.2:**

Suppose  $\sum_{i \in \mathcal{I}} (\bar{x}^i - e^i) \neq 0$ . From Assertion 4.2 we have

$$\forall p \in B, \quad p \square \sum_{i \in \mathcal{I}} (\bar{x}^i - e^i) \leq \bar{p} \square \sum_{i \in \mathcal{I}} (\bar{x}^i - e^i)$$

scalar product with  $\lambda$  gives

$$\forall p \in B, \quad (\lambda \square p) \bullet_{\mathbb{L}} \sum_{i \in \mathcal{I}} (\bar{x}^i - e^i) \leq (\lambda \square \bar{p}) \bullet_{\mathbb{L}} \sum_{i \in \mathcal{I}} (\bar{x}^i - e^i)$$

Thus  $(\lambda \square \bar{p}) = \frac{\sum_{i \in \mathcal{I}} (\bar{x}^i - e^i)}{\|\sum_{i \in \mathcal{I}} (\bar{x}^i - e^i)\|}$  and  $\|\lambda \square p\| = 1$ . So  $(\lambda \square \bar{p}) \bullet_{\mathbb{L}} \sum_{i \in \mathcal{I}} (\bar{x}^i - e^i) > 0$ .

From Assertion (4.3)  $\forall i \in \mathcal{I}, \exists \bar{\tau}_i \in [0, 1]$  such that

$$\bar{p} \square (\bar{x}^i - e^i) \leq \bar{w}^i + \bar{\tau}_i \gamma(\bar{p}) + \rho(\bar{p}) \mathbf{1}$$

sum over  $i$  to get:

$$\bar{p} \square \sum_{i \in \mathcal{I}} (\bar{x}^i - e^i) \leq \sum_{i \in \mathcal{I}} \bar{w}^i + \left( \sum_{i \in \mathcal{I}} \bar{\tau}_i \right) \gamma(\bar{p}) + I \rho(\bar{p}) \mathbf{1}$$

taking scalar product with  $\lambda$  we have the following:

$$(\lambda \square \bar{p}) \bullet_{\mathbb{L}} \sum_{i \in \mathcal{I}} (\bar{x}^i - e^i) \leq \lambda \bullet_{\mathbb{D}} \sum_{i \in \mathcal{I}} \bar{w}^i + \lambda \bullet_{\mathbb{D}} \left( \sum_{i \in \mathcal{I}} \bar{\tau}_i \right) \gamma(\bar{p}) + I \rho(\bar{p}) \lambda \bullet \mathbf{1}$$

On the RHS we have

$$\lambda \bullet_{\mathbb{D}} \sum_{i \in \mathcal{I}} \bar{w}^i = 0 \quad (\text{ by Lemma 4.5.1})$$

$$\left( \sum_{i \in \mathcal{I}} \bar{\tau}_i \right) \lambda \bullet_{\mathbb{D}} \gamma(\bar{p}) = 0 \quad (\text{ since } \gamma \in \Gamma_1)$$

$$\rho(\bar{p}) = 0 \quad (\text{ since } \|\lambda \square \bar{p}\| = 1)$$

Thus  $(\lambda \square \bar{p}) \bullet_{\mathbb{L}} \sum_{i \in \mathcal{I}} (\bar{x}^i - e^i) \leq 0$ . Contradiction.  $\square$

**Claim 4.5.3** *The following conditions hold:*

$$(i) \quad \forall \xi \in \mathbb{R}^{\mathbb{D}}, \bar{p}(\xi) \neq 0$$

$$(ii) \forall i \in \mathcal{I}, (\bar{x}^i, \bar{w}^i) \in \beta^i(\bar{p}) \text{ and } \beta^i(\bar{p}) \cap (P^i(\bar{x}) \times WZ^i) = \emptyset$$

**Proof of Claim 4.5.3:** Since  $\gamma \in \Gamma_3, \exists k \in \mathcal{I}$  such that  $\alpha^k(\bar{p}) \neq \emptyset$ . We will first show Condition (ii) for consumer  $k$ .

From Assertion (4.3) in the fixed point theorem, for consumer  $k$  one has  $(\bar{x}^k, \bar{w}^k) \in \beta^k(\bar{p})$ . Now, suppose that  $\beta^k(\bar{p}) \cap (P^k(\bar{x}) \times WZ^k) \neq \emptyset$  and let  $(x^k, w^k) \in \beta^k(\bar{p}) \cap (P^k(\bar{x}) \times WZ^k)$ . Since  $\gamma \in \Gamma_3$  we have  $\alpha^k(\bar{p}) \neq \emptyset$  and we let  $(\bar{x}^k, \bar{w}^k) \in \alpha^k(\bar{p})$ .

Suppose first that  $\bar{x}^k = x^k$ , then, from above  $(x^k, \bar{w}^k) \in [P^k(\bar{x}) \times WZ^k] \cap \alpha^k(\bar{p})$ , which contradicts the fact that this set is empty by Assertion (4.3). Suppose now that  $\bar{x}^k \neq x^k$ , from Assumption (C.iii), (recalling that  $x^k \in P^k(\bar{x})$ ) the set  $[\bar{x}^k, x^k[ \cap P^k(\bar{x})$  is nonempty, hence contains a point  $x^k(\lambda) := (1 - \lambda)\bar{x}^k + \lambda x^k$  for some  $\lambda \in [0, 1[$ . We let  $w^k(\lambda) := (1 - \lambda)\bar{w}^k + \lambda w^k$  and we check that  $(x^k(\lambda), w^k(\lambda)) \in \alpha^k(\bar{p})$  (since  $(x^k, w^k) \in \beta^k(\bar{p})$  and  $(\bar{x}^k, \bar{w}^k) \in \alpha^k(\bar{p})$ ). Consequently,  $\alpha^k(\bar{p}) \cap (P^k(\bar{x}) \times WZ^k) \neq \emptyset$ , which contradicts again Assertion (4.3).

Thus for agent  $k$  we have

$$(\bar{x}^k, \bar{w}^k) \in \beta^k(\bar{p}) \text{ and } \beta^k(\bar{p}) \cap (P^k(\bar{x}) \times WZ^k) = \emptyset \quad (**)$$

**Proof of (i):** Suppose there exists  $\xi \in \mathcal{D}$  such that  $p(\xi) = 0$ . From Claim 4.5.2,  $\sum_{i \in \mathcal{I}} \bar{x}^i = \sum_{i \in \mathcal{I}} e^i$ , and from the Non-Satiation Assumption at node  $\xi$  (for Consumer  $k$ ) there exists  $x^k \in P^k(\bar{x})$  such that  $x^k(\xi') = \bar{x}^k(\xi')$  for every  $\xi' \neq \xi$ ; from Assertion (4.3),  $(\bar{x}^k, \bar{w}^k) \in \beta^k(\bar{p})$  and, recalling that  $\bar{p}(\xi) = 0$ , one deduces that  $(x^k, \bar{w}^k) \in \beta^k(\bar{p})$ . Consequently,

$$\beta^k(\bar{p}) \cap [P^k(\bar{x}) \cap WZ^k] \neq \emptyset,$$

which contradicts (\*\*).

**Proof of (ii):** From Assertion (4.3), for all  $i \in \mathcal{I}$  one has  $(\bar{x}^i, \bar{w}^i) \in \beta^i(\bar{p})$ . From the Survival Assumption and the fact that  $\bar{p}(\xi) \neq 0$  for every  $\xi \in \mathcal{D}$  (by Part (i) of this claim), one deduces that  $\alpha^i(\bar{p}) \neq \emptyset$ .<sup>6</sup>

For each  $i \in \mathcal{I} \setminus \{k\}$  we can repeating the same steps done in the beginning of the proof of this claim to get that for all  $i \in \mathcal{I}$ ,  $(\bar{x}^i, \bar{w}^i) \in \beta^i(\bar{p})$  and  $\beta^i(\bar{p}) \cap (P^i(\bar{x}) \times WZ^i) = \emptyset$ .  $\square$

**Claim 4.5.4** *The following conditions hold:*

$$(i) \quad \rho(\bar{p}) = 0$$

$$(ii) \quad \sum_{i \in \mathcal{I}} \bar{w}^i = 0$$

$$(iii) \quad \forall i \in \mathcal{I}, \bar{\tau}^i \gamma(\bar{p}) = 0$$

**Proof of Claim 4.5.4:**

**Proof of (i):** We first prove that the modified budget constraints are binding, that is for all  $i \in \mathcal{I}$  we have the following assertion:

$$\bar{p} \square (\bar{x}^i - e^i) = \bar{w}^i + \bar{\tau}^i \gamma(\bar{p}) + \rho(\bar{p}) \mathbf{1} \quad (4.5)$$

Suppose not, then there exists  $i \in \mathcal{I}$  such that

$$\bar{p} \square (\bar{x}^i - e^i) < \bar{w}^i + \bar{\tau}^i \gamma(\bar{p}) + \rho(\bar{p}) \mathbf{1}$$

That is there exist  $\xi \in \mathcal{D}$  such that

$$\bar{p}(\xi) \bullet_{\mathbb{H}} (\bar{x}^i(\xi) - e^i(\xi)) < \bar{w}^i(\xi) + \bar{\tau}^i \gamma(\bar{p})(\xi) + \rho(\bar{p})$$

---

<sup>6</sup>Take  $\bar{w}^i = 0, \bar{\tau}^i = 0$  and  $\bar{x}^i = e^i - t\bar{p}$  for  $t > 0$  small enough, so that  $\bar{x}^i \in X^i$  (from the Survival Assumption). Then notice that  $\bar{p} \square (\bar{x}^i - e^i) = -t(\bar{p} \square \bar{p}) \ll 0 \leq +0 + \rho(\bar{p})$ .

But by Claim 4.5.2,  $\sum_{i \in \mathcal{I}} \bar{x}^i = \sum_{i \in \mathcal{I}} e^i$  and by the nonsatiation assumption C(v) for consumer  $i$ , there exists  $x^i \in P^i(\bar{x})$  such that  $x^i(\xi') = \bar{x}^i(\xi')$  for every  $\xi' \neq \xi$ . Consequently, we can choose  $x \in [x^i, \bar{x}^i[$  close enough to  $\bar{x}^i$  so that  $(x, \bar{w}^i) \in \beta^i(\bar{p})$ . But, from the local non-satiation (Assumption (K.ii)),  $[x^i, \bar{x}^i[ \subset P^i(\bar{x})$ . Consequently,  $\beta^i(\bar{p}) \cap (P^i(\bar{x}) \times WZ^i) \neq \emptyset$  which contradicts Claim 4.5.3.

In Assertion 4.5 taking scalar product with  $\lambda$  yields:

$$(\lambda \square \bar{p}) \bullet_{\mathbb{L}} (\bar{x}^i - e^i) = \lambda \bullet_{\mathbb{D}} \bar{w}^i + \bar{\tau}^i \lambda \bullet_{\mathbb{D}} \gamma(\bar{p}) + \rho(\bar{p}) \lambda \bullet_{\mathbb{L}} \mathbf{1}$$

Since  $\gamma \in \Gamma_1$ ,  $\lambda \bullet_{\mathbb{D}} \gamma(\bar{p}) = 0$ . So we have,

$$(\lambda \square \bar{p}) \bullet_{\mathbb{H}} (\bar{x}^i - e^i) = \lambda \bullet_{\mathbb{D}} \bar{w}^i + \rho(\bar{p}) \lambda \bullet_{\mathbb{D}} \mathbf{1}$$

Summing over  $i \in \mathcal{I}$ , we have

$$(\lambda \square \bar{p}) \bullet_{\mathbb{L}} \left( \sum_{i \in \mathcal{I}} (\bar{x}^i - e^i) \right) = \lambda \bullet_{\mathbb{D}} \left( \sum_{i \in \mathcal{I}} \bar{w}^i \right) + I \rho(\bar{p}) \lambda \bullet_{\mathbb{D}} \mathbf{1}$$

Since  $\sum_{i \in \mathcal{I}} \bar{w}^i \in W(\bar{q})(\sum_{i \in \mathcal{I}} Z^i)$ , by Lemma 4.5.1 and Claim 4.5.2 we have:

$$0 = \mathbb{I} \rho(\bar{p}) \lambda \bullet_{\mathbb{D}} \mathbf{1}$$

Thus  $\rho(\bar{p}) = 0$ .  $\square$

**Proof of (ii) and (iii):** In Assertion (4.5) in view of Claim 4.5.4 (i), we have  $\forall i \in \mathcal{I}$ :

$$\bar{p} \square (\bar{x}^i - e^i) = \bar{w}^i + \bar{\tau}^i \gamma(\bar{p})$$

Summing over  $i \in \mathcal{I}$  and using Claim 4.5.2 we have:

$$0 = \sum_{i \in \mathcal{I}} \bar{w}^i + \left( \sum_{i \in \mathcal{I}} \bar{\tau}^i \right) \gamma(\bar{p})$$

Since  $\gamma \in \Gamma_2$  we have

$$0 = \left( \sum_{i \in \mathcal{I}} \bar{\tau}^i \right) \gamma(\bar{p}) \cdot \left( \sum_{i \in \mathcal{I}} \bar{w}^i \right) = - \left( \sum_{i \in \mathcal{I}} \bar{w}^i \right) \cdot \left( \sum_{i \in \mathcal{I}} \bar{w}^i \right)$$

Thus  $\sum_{i \in \mathcal{I}} \bar{w}^i = 0$ . Since  $\forall i \in \mathcal{I}, \bar{\tau}^i \geq 0$  and  $(\sum_{i \in \mathcal{I}} \bar{\tau}^i) \gamma(\bar{p}) = 0$  we have  $\forall i \in \mathcal{I}, \bar{\tau}^i \gamma(\bar{p}) = 0$ .  $\square$

**Claim 4.5.5** *For every  $i \in \mathcal{I}$ ,  $(\bar{x}^i, \bar{z}^i) \in B^i(\bar{p}, \bar{q})$  and  $[P^i(\bar{x}) \times Z^i] \cap B^i(\bar{p}, \bar{q}) = \emptyset$ .*

**Proof of Claim 4.5.5:** In view of Claim 4.5.4 and Assertion 4.3, we have for all  $i \in \mathcal{I}$

$$p \square (\bar{x}^i - e^i) = \bar{w}^i$$

Thus in view of Assertion 4.4  $(\bar{x}^i, \bar{z}^i) \in B^i(\bar{p}, \bar{q})$ .

Moreover by Assertion 4.3

$$\beta^i(\bar{p}) \cap [P^i(\bar{x}) \times WZ^i] = \emptyset$$

This condition along with Assertion 4.4 implies

$$B^i(\bar{p}, \bar{q}) \cap [P^i(\bar{x}) \times Z^i] = \emptyset$$

Note that by Assertion 4.4 for all  $i \in \mathcal{I}$  there exists  $\bar{z}^i \in Z^i$  such that  $\bar{w}^i = W(\bar{q})\bar{z}^i$ . Suppose  $B^i(\bar{p}, \bar{q}) \cap [P^i(\bar{x}) \times Z^i] \neq \emptyset$ . Then it contains a point  $(x^i, z^i)$  such that

$$p \square (x^i - e^i) \leq W(\bar{q})z^i$$

Letting  $w^i = W(\bar{q})z^i$  we have  $(x^i, w^i) \in \beta^i(\bar{p})$ . Also  $x^i \in P^i(\bar{x})$ . Thus  $(x^i, w^i) \in [P^i(\bar{x}) \times WZ^i]$ . Which implies  $\beta^i(\bar{p}) \cap [P^i(\bar{x}) \times WZ^i] \neq \emptyset$ . Contradiction with Assertion 4.3.]  $\square$

In view of Claim 4.5.2, Assertion 4.4, Claim 4.5.3, Claim 4.5.4 and Claim 4.5.5 we have  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  is a weak equilibrium.

### 4.5.2 Proof in the general case (without additional assumptions)

We now give the proof of Theorem 4.2.2, without considering the additional Assumption (K), as in the previous section. We will first enlarge the strict preferred sets as in Gale-Mas Colell, and then truncate the economy  $\mathcal{E}$  by a standard argument to define a new economy  $\hat{\mathcal{E}}_r$ , which satisfies all the assumptions of  $\mathcal{E}$ , together with the additional Assumption (K). From the previous section, there exists a weak equilibrium of  $\hat{\mathcal{E}}_r$  and we will then check that it is also a weak equilibrium of  $\mathcal{E}$ .

#### Enlarging the preferences as in Gale and Mas-Colell

The original preferences  $P^i$  are replaced by the "enlarged" ones  $\hat{P}^i$  defined as follows. For every  $i \in \mathcal{I}$ ,  $\bar{x} \in \prod_{i \in \mathcal{I}} X^i$  we let

$$\hat{P}^i(\bar{x}) := \bigcup_{x^i \in P^i(\bar{x})} ]\bar{x}^i, x^i] = \{\bar{x}^i + t(x^i - \bar{x}^i) \mid t \in ]0, 1], x^i \in P^i(\bar{x})\}.$$

The next proposition shows that  $\hat{P}^i$  satisfies the same properties as  $P^i$ , for every  $i \in \mathcal{I}$ , together with the additional Local Nonsatiation Assumption (K.ii).

**Proposition 4.5.1** *Under (C), for every  $i \in \mathcal{I}$  and every  $\bar{x} \in \prod_{i \in \mathcal{I}} X^i$  one has:*

- (i)  $P^i(\bar{x}) \subset \hat{P}^i(\bar{x}) \subset X^i$ ;
- (ii) *the correspondence  $\hat{P}^i$  is lower semicontinuous at  $\bar{x}$  and  $\hat{P}^i(\bar{x})$  is convex;*
- (iii) *for every  $y^i \in \hat{P}^i(\bar{x})$  for every  $(x')^i \in X^i$ ,  $(x')^i \neq y^i$  then  $[(x')^i, y^i[ \cap \hat{P}^i(\bar{x}) \neq \emptyset$ ;*



(iv)  $\bar{x}^i \notin \hat{P}^i(\bar{x})$ ;

(v) (*Non-Satiation at Every Node*) if  $\sum_{i \in \mathcal{I}} \bar{x}^i = \sum_{i \in \mathcal{I}} e^i$ , for every  $\xi \in \mathcal{D}$ , there exists  $x \in \prod_{i \in \mathcal{I}} X^i$  such that, for each  $\xi' \neq \xi$ ,  $x^i(\xi') = \bar{x}^i(\xi')$  and  $x^i \in \hat{P}^i(\bar{x})$ ;

(vi) for every  $y_i \in \hat{P}^i(\bar{x})$ , then  $[y^i, \bar{x}^i] \subset \hat{P}^i(\bar{x})$ .

**Proof.** Let  $\bar{x} \in \prod_{i \in \mathcal{I}} X^i$  and let  $i \in \mathcal{I}$ .

*Part (i).* It follows by the convexity of  $X^i$ , for every  $i \in \mathcal{I}$ .

*Part (ii).* Let  $y^i \in \hat{P}^i(\bar{x})$  and consider a sequence  $(\bar{x}_n)_n \subset \prod_{i \in \mathcal{I}} X^i$  converging to  $\bar{x}$ . Since  $y^i \in \hat{P}^i(\bar{x})$ , then  $y^i = \bar{x}^i + t(x^i - \bar{x}^i)$  for some  $x^i \in P^i(\bar{x})$  and some  $t \in ]0, 1]$ . Since  $P^i$  is lower semicontinuous, there exists a sequence  $(x_n^i)$  converging to  $x^i$  such that  $x_n^i \in P^i(\bar{x}_n)$  for every  $n \in \mathbb{N}$ . Now define  $y_n^i := \bar{x}_n^i + t(x_n^i - \bar{x}_n^i) \in ]\bar{x}_n^i, x_n^i]$ : then  $y_n^i \in \hat{P}^i(\bar{x}_n)$  and obviously the sequence  $(y_n^i)$  converges to  $y^i$ . This shows that  $\hat{P}^i$  is lower semicontinuous at  $\bar{x}$ .

To show that  $\hat{P}^i(\bar{x})$  is convex, let  $y_1^i, y_2^i \in \hat{P}^i(\bar{x})$ , let  $\lambda_1 \geq 0, \lambda_2 \geq 0$ , such that  $\lambda_1 + \lambda_2 = 1$ , we show that  $\lambda_1 y_1^i + \lambda_2 y_2^i \in \hat{P}^i(\bar{x})$ . Then  $y_k^i = \bar{x}^i + t_k(x_k^i - \bar{x}^i)$  for some  $t_k \in ]0, 1]$  and some  $x_k^i \in P^i(\bar{x})$  ( $k = 1, 2$ ). One has

$$\lambda_1 y_1^i + \lambda_2 y_2^i = \bar{x}^i + (\lambda_1 t_1 + \lambda_2 t_2)(x^i - \bar{x}^i),$$

where  $x^i := (\lambda_1 t_1 x_1^i + \lambda_2 t_2 x_2^i) / (\lambda_1 t_1 + \lambda_2 t_2) \in P^i(\bar{x})$  (since  $P^i(\bar{x})$  is convex, by Assumption **(C)**) and  $\lambda_1 t_1 + \lambda_2 t_2 \in ]0, 1]$ . Hence  $\lambda_1 y_1^i + \lambda_2 y_2^i \in \hat{P}^i(\bar{x})$ .

*Part (iii).* Let  $y^i \in \hat{P}^i(\bar{x})$  and let  $(x')^i \in X^i, (x')^i \neq y^i$ . From the definition of  $\hat{P}^i$ ,  $y^i = \bar{x}^i + t(x^i - \bar{x}^i)$  for some  $x^i \in P^i(\bar{x})$  and some  $t \in ]0, 1]$ . Suppose first that  $x^i = (x')^i$ , then  $y^i \in ]\bar{x}^i, x^i] \subset \hat{P}^i(\bar{x})$ . Consequently,  $[(x')^i, y^i] \cap \hat{P}^i(\bar{x}) \neq \emptyset$ . Suppose now that  $x^i \neq (x')^i$ ; since  $P^i$  satisfies Assumption *(C.iii)*, there exists  $\lambda \in [0, 1[$  such that  $x^i(\lambda) = (x')^i + \lambda(x^i - (x')^i) \in P^i(\bar{x})$ . We let

$$z := [\lambda(1 - t)\bar{x}^i + t(1 - \lambda)(x')^i + t\lambda x^i] / \alpha \quad \text{with } \alpha := t + \lambda(1 - t),$$

and we check that  $z = [\lambda(1-t)\bar{x}^i + tx^i(\lambda)]/\alpha \in ]\bar{x}^i, x^i(\lambda)[$ , with  $x^i(\lambda) \in P^i(\bar{x})$ , hence  $z \in \hat{P}^i(\bar{x})$ . Moreover,  $z := [\lambda y^i + t(1-\lambda)(x')^i]/\alpha \in [(x')^i, y^i[$ . Consequently,  $[(x')^i, y^i] \cap \hat{P}^i(\bar{x}) \neq \emptyset$ , which ends the proof of (iii).

*Parts (iv), (v) and (vi).* They follow immediately by the definition of  $\hat{P}^i$  and the properties satisfied by  $P^i$  in **(C)**.  $\square$

## Truncating the economy

Given  $q \in \mathbb{R}^{\mathbb{J}}$  the set of admissible consumptions and income transfers,  $K(q)$  is given by:

$$K(q) := \{(x, w) \in \prod_{i \in \mathcal{I}} X^i \times \prod_{i \in \mathcal{I}} W(q)Z^i : \exists p \in B_L(0, 1), \\ (x^i, w^i) \in B^i(p, q) \text{ for every } i \in \mathcal{I}, \sum_{i \in \mathcal{I}} x^i = \sum_{i \in \mathcal{I}} e^i, \sum_{i \in \mathcal{I}} w^i = 0\}.$$

**Lemma 4.5.3**  *$K(q)$  is bounded.*

**Proof of Lemma 4.5.3:** Given  $q \in \mathbb{R}^{\mathbb{J}}$ , for every  $i \in \mathcal{I}$  define the following:

$$\hat{X}^i(q) := \left\{ x^i \in X^i : \exists (x^j)_{j \neq i} \in \prod_{j \neq i} X^j, \exists w \in \prod_{i \in \mathcal{I}} WZ^i, (x, w) \in K(q) \right\}$$

and

$$\widehat{W}^i(q) := \left\{ w^i \in WZ^i : \exists (w^j)_{j \neq i} \in \prod_{j \neq i} WZ^j, \exists x \in \prod_{i \in \mathcal{I}} X^i, (x, w) \in K(q) \right\}.$$

We need to show that  $\hat{X}^i(q)$  and  $\widehat{W}^i(q)$  are bounded. Since  $\hat{X}^i$  is compact (by Assumption C (i)), clearly  $\hat{X}^i(q)$  is bounded. To show  $\widehat{W}^i(q)$  is bounded, let  $w^i \in \widehat{W}^i(q)$ . Since

$$(x^i, w^i) \in \{(x, w) \in X^i \times W(q)Z^i \mid p \square (x - e^i) \leq w\}$$

and  $(x^i, p) \in \hat{X}^i(q) \times B_{\mathbb{L}}(0, 1)$ , a compact set from above,  $\exists \alpha^i \in \mathbb{R}^{\mathbb{D}}$ , such that

$$\alpha^i \leq p \square (x^i - e^i) \leq w^i$$

Using the fact that  $\sum_{i \in \mathcal{I}} w^i = 0$  we also have

$$w^i = - \sum_{j \neq i} w^j \leq - \sum_{j \neq i} \alpha^j,$$

Thus  $\widehat{W}^i(q)$  is bounded for every  $i \in \mathcal{I}$ .  $\square$

We now define the "truncated economy" as follows.

Since  $\hat{X}^i(q)$  and  $\hat{W}^i(q)$  are bounded subsets of  $\mathbb{R}^{\mathbb{L}}$  and  $\mathbb{R}^{\mathbb{D}}$ , respectively (by Lemma 4.5.3), there exists a real number  $r > 0$  such that, for every agent  $i \in \mathcal{I}$ ,  $\hat{X}^i(\bar{q}) \subset \text{int} B_{\mathbb{L}}(0, r)$  and  $\hat{W}^i(\bar{q}) \subset \text{int} B_{\mathbb{D}}(0, r)$ . The truncated economy  $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$  is the collection

$$(\hat{\mathcal{E}}_r, \mathcal{F}_r) = [\mathcal{D}, \mathcal{H}, \mathcal{I}, (X_r^i, \hat{P}_r^i, e^i)_{i \in \mathcal{I}}; \mathcal{J}, (\xi(j), V^j)_{j \in \mathcal{J}}, (Z_r^i)_{i \in \mathcal{I}}],$$

where,

$$X_r^i = X^i \cap B_{\mathbb{L}}(0, r), \quad Z_r^i = \{z \in Z^i \mid W(\bar{q})z \in B_{\mathbb{D}}(0, r)\} \text{ and } \hat{P}_r^i(x) = \hat{P}^i(x) \cap \text{int} B_{\mathbb{L}}(0, r).$$

The existence of a weak equilibrium of  $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$  is then a consequence of Section 4.5.1, that is, Theorem 4.2.2 with the additional Assumption (K). We just have to check that Assumption (K) and all the assumptions of Theorem 4.2.2 are satisfied by  $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$ . In view of Proposition 4.5.1, this is clearly the case for all the assumptions but the Survival Assumption (C.vi) that is proved via a standard argument (that we recall hereafter).

Indeed we first notice that  $(e^i, 0)_{i \in \mathcal{I}}$  belongs to  $K(q)$ , hence, for every  $i \in \mathcal{I}$ ,  $e^i \in \hat{X}^i(q) \subset \text{int} B_{\mathbb{L}}(0, r)$ . Recalling that  $e^i \in \text{int} X^i$  (from the Survival Assumption), we deduce that  $e^i \in \text{int} X^i \cap \text{int} B_{\mathbb{L}}(0, r) \subset \text{int}[X^i \cap B_{\mathbb{L}}(0, r)] = \text{int} X_r^i$ .

**Proposition 4.5.2** *Given  $\bar{q} \in \mathbb{R}^J$ , if  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  is a weak equilibrium of  $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$  then it is also a weak equilibrium of  $(\mathcal{E}, \mathcal{F})$ .*

**Proof of Proposition 4.5.2.** Let  $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$  be a weak equilibrium of the economy  $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$ . In view of the definition of a weak equilibrium, to prove that it is also a weak equilibrium of  $(\mathcal{E}, \mathcal{F})$  we only have to check, for every  $i \in \mathcal{I}$ ,  $[P^i(\bar{x}) \times Z^i] \cap B^i(\bar{p}, \bar{q}) = \emptyset$ , where  $B^i(\bar{p}, \bar{q})$  denotes the budget set of agent  $i$  in the economy  $(\mathcal{E}, \mathcal{F})$ .

Assume, on the contrary, that, for some  $i \in \mathcal{I}$  the set  $[P^i(\bar{x}) \times Z^i] \cap B^i(\bar{p}, \bar{q})$  is nonempty, hence contains a couple  $(x^i, z^i)$ . Clearly the allocation  $(\bar{x}, W(\bar{q})\bar{z})$  belongs to the set  $K(\bar{q})$ , hence for every  $i \in \mathcal{I}$ ,  $\bar{x}^i \in \hat{X}^i(\bar{q}) \subset \text{int} B_{\mathbb{L}}(0, r)$  and  $\bar{w}^i = W(\bar{q})\bar{z}^i \in \hat{W}^i(\bar{q}) \subset \text{int} B_{\mathbb{D}}(0, r)$ . Thus, for  $t \in ]0, 1]$  sufficiently small,  $x^i(t) := \bar{x}^i + t(x^i - \bar{x}^i) \in \text{int} B_{\mathbb{L}}(0, r)$  and  $w^i(t) := \bar{w}^i + t(w^i - \bar{w}^i) \in \text{int} B_{\mathbb{D}}(0, r)$ . Clearly  $(x^i(t), w^i(t))$  is such that  $w^i(t) = W(\bar{q})z^i(t)$  where  $z^i(t) = (\bar{z}^i + t(z^i - \bar{z}^i)) \in Z^i$  and  $(x^i(t), z^i(t))$  belongs to the budget set  $B^i(\bar{p}, \bar{q})$  of agent  $i$  (for the economy  $(\mathcal{E}, \mathcal{F})$ ) and since  $x^i(t) \in X_r^i := X^i \cap B_{\mathbb{L}}(0, r)$ ,  $z^i(t) \in Z_r^i := \{z \in Z^i \mid W(\bar{q})z \in B_{\mathbb{D}}(0, r)\}$ , the couple  $(x^i(t), z^i(t))$  belongs also to the budget set  $B_r^i(\bar{p}, \bar{q})$  of agent  $i$  (in the economy  $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$ ). From the definition of  $\hat{P}^i$ , we deduce that  $x^i(t) \in \hat{P}^i(\bar{x})$  (since from above  $x^i(t) := \bar{x}^i + t(x^i - \bar{x}^i)$  and  $x^i \in P^i(\bar{x})$ ), hence  $x^i(t) \in \hat{P}_r^i(\bar{x}) := \hat{P}^i(\bar{x}) \cap \text{int} B_{\mathbb{L}}(0, r)$ . We have thus shown that, for  $t \in ]0, 1]$  small enough,  $(x^i(t), z^i(t)) \in [\hat{P}_r^i(\bar{x}) \times Z_r^i] \cap B_r^i(\bar{p}, \bar{q})$ . This contradicts the fact that this set is empty, since  $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$  is a weak equilibrium of the economy  $(\hat{\mathcal{E}}_r, \mathcal{F}_r)$ .  $\square$

## Chapter 5

### Conclusion

Restricted participation in financial markets is more of a norm than exception than an exception. It is thus useful and important to understand the implications of this phenomenon in a general equilibrium framework. A complete and satisfactory analysis of the implication of such restrictions should in principle consider these restrictions as arising endogenously. However the first and necessary procedure without getting into the institutional reasons behind these restrictions would be of impose these restrictions as a primitive describing the agents. This procedure has been adopted in this thesis.

Limits on the capabilities of agents to process information in timely manner may force agents to concentrate on a subset of the available assets. This is equivalent to imposing a very special type of linear restrictions on the portfolio sets of agents. This was the framework in Chapter 3, of this thesis. once these restrictions have been imposed, the notion of arbitrage and its absence differs at the individual and aggregate levels. Chapter 3, presented this difference in a multi-period setting. The central result states that generically the set of asset prices the do not offer arbitrage to each agent individually will be larger than those prices that preclude arbitrage at the collective or aggregate level.

A natural extension to this result would be to consider the case where each agent's portfolio set is a strict subspace of the space of all possible portfolios. This could be associated with the situation where agent are allowed to invest only through some mutual fund. I believe this extension should arise naturally, however it remains to be proved.

The consequences of the difference in the notions of arbitrage at the individual and collective level in a general equilibrium setting is the subject matter of Chapter 4. Owing to the fact that the set of prices that do not offer arbitrage at the individual level may be too large, we proceed to show that the set of prices that do not offer arbitrage at the aggregate level may also be equilibrium prices. However this characterization will be possible only under some conditions of the linear space spanned by the set of payoffs generated by the aggregation of portfolio sets - the transfer space condition. The motivation for the condition given in this paper (which generalizes many of the previous conditions in the literature) arises from the conjecture of Cass [13], which was used in the linear restrictions case.

## 5.1 Extensions and future research

I believe, there is still some work that is needed in completely exploring the characterization, of equilibrium asset prices with the the asset prices that do not offer arbitrage at the aggregate level. The counter examples in Chapter 4 work with carefully chosen endowment and preferences. Thus the robustness of this characterization, needs to be studied in a more abstract setting.

Another confounding yet interesting result from Chapter 4 is that in order to weaken the definition of absence of arbitrage that is being consider for

the central characterization result, the corresponding transfer space condition must be strengthened. This indicates that there are some crucial interactions between the arbitrage opportunities and the income transfer possibilities, that arise at the individual level versus those that arise at the aggregate level. This issue I believe deserves further exploration in its own right. Only then can we truly understand the implications of such restricted participation on asset markets.

One other interesting and necessary line of inquiry would be to consider these questions in a model with real assets. In all the analysis of this thesis, all assets were nominal. This enabled us to analyze the financial aspect and real aspects of the economy separately, since the asset payoffs do not depend on spot prices. However with real assets this simplification is not possible and could yield interesting results.

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